Preliminary Exam: Quantum Mechanics, Thursday August 27, 2020. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. If a student submits solutions to more than four problems, only the first four problems as listed on the exam will be graded. Students should write their solutions on blank 8.5 by 11 paper or in a blue book, putting their name on each page, the number of the problem and the number of the page in their solution on each page (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their solutions in sequence using a cell phone or a scanner and email them in a file or files to the prelim committee chair philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam. (It might be easier to transfer the files to a laptop first.) Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems. The chair will immediately check if the emailing is readable or if a resend is required.

Some possibly useful information

$$\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\\ &= \frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}\end{aligned}$$

$$\boldsymbol{\nabla}\psi = \mathbf{e}_x\frac{\partial\psi}{\partial x}+\mathbf{e}_y\frac{\partial\psi}{\partial y}+\mathbf{e}_z\frac{\partial\psi}{\partial z} = \mathbf{e}_r\frac{\partial\psi}{\partial r}+\mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial \theta}+\mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} = \mathbf{e}_\rho\frac{\partial\psi}{\partial\rho}+\mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}+\mathbf{e}_z\frac{\partial\psi}{\partial z}.$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \;,\quad H_0(x)=1 \;,\quad H_1(x)=2x \;,\quad H_2(x)=4x^2-2$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right) \;,\quad \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r) \;.$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \;,\quad P_0(x)=1 \;,\quad P_1(x)=x \;,\quad P_2(x)=\frac{1}{2}(3x^2-1) \;,$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi} \;,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \;,\quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \;,\quad Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1) \;,\quad Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \;,\quad Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels :}\quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \;,\quad n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \;,$$

$$\text{with asymptotic behavior}\quad j_\ell(r)\to\frac{\cos(r-\ell\pi/2-\pi/2)}{r} \;,\quad n_\ell(r)\to\frac{\sin(r-\ell\pi/2-\pi/2)}{r} \;.$$

$$j_0(r) = \frac{\sin r}{r} \;,\quad n_0(r) = -\frac{\cos r}{r} \;,\quad j_1(r) = \frac{\sin r}{r^2}-\frac{\cos r}{r} \;,\quad n_1(r) = -\frac{\cos r}{r^2}-\frac{\sin r}{r} \;,$$

$$j_2(r) = \frac{3\sin r}{r^3}-\frac{\sin r}{r}-\frac{3\cos r}{r^2} \;,\quad n_2(r) = -\frac{3\cos r}{r^3}+\frac{\cos r}{r}-\frac{3\sin r}{r^2} \;.$$

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta).$$

1. **(a)** Eigenstates of the (non-Hermitian) lowering operator a are defined as coherent (or squeezed) states in the one-dimensional harmonic oscillator problem. Show that in terms of the raising operator $a^\dagger$ these states can be written as $\hat{S}(\alpha)\,|0\rangle$ where

$$\hat{S}(\alpha) = N(\alpha)e^{\alpha a^\dagger}$$

and find the constant $N(\alpha)$ that would normalize the states.

(b) Show that these coherent states give rise to the minimum uncertainty product in position and momentum.

(c) Establish whether or not a similar set of eigenstates can be obtained for the raising operator $a^\dagger$.

2. **(a)** Three spinless point masses each of mass m, obeying Bose statistics, are confined to move on a circle of radius R. Their mutual distances are equal and they form an equilateral triangle. What is the energy spectrum and degeneracy factor for each energy level of this system.

(b) Two identical spin 1/2 particles obeying Fermi statistics are confined to a cubic box whose sides are L in length. There exists an attractive potential between the pair of particles of strength V_0, acting whenever the distance between them is less than $d \ll 2L$, so that you may approximate this interaction as

$$V(\mathbf{r}_1 - \mathbf{r}_2) \simeq -\left(\frac{4\pi V_0 d^3}{3}\right)\delta^3(\mathbf{r}_1 - \mathbf{r}_2).$$

In order to use non-relativistic perturbation theory, treat the unperturbed system as the case when the particles are noninteracting.

(i) Find the unperturbed ground state wavefunction of the two-particle system including both the spatial and spin degrees of freedom of the two particles.

(ii) Find the ground state energy of the system using first-order perturbation theory.

3. The ground-state wave-function of a non-relativistic particle with mass m in a one-dimensional quantum system is

$$\psi_0(x,t) = (L - |x|)\,e^{-i\lambda t/\hbar} \quad \text{if } |x| < L, \qquad \psi_0(x,t) = 0 \quad \text{if } |x| > L.$$

Here $L > 0$ and λ is real.

(a) Determine the Hamiltonian operator of this quantum system.

(b) Compute the expectation values of the kinetic energy and potential energy in the state $\psi_0(x,t)$.

(c) Compute the expectation values $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$ in the state $\psi_0(x,t)$.

(d) Investigate whether in the state $\psi_0(x,t)$ the system obeys the uncertainty relation $\Delta x\,\Delta p \geq \frac{1}{2}\hbar$.

4. In the non-relativistic limit the ground state of a point-like spin zero particle of mass m and charge $(-e)$ in the Coulomb potential V_{Coul} of a point-like infinitely heavy spin-zero nucleus of charge Ze is described by the wave-function

$$\Phi_0(\vec{x}, t) = \frac{1}{\sqrt{\pi\, a^3}}\, \exp(-\frac{r}{a})\, e^{-iE_0\, t/\hbar}$$

where $E_0 = -Z^2 \frac{\hbar^2}{2m\, a_B^2}$, $a = a_B/Z$ and $a_B = \frac{4\pi\varepsilon_0 \hbar^2}{e^2 m}$.

(a) The kinetic energy operator in the non-relativistic Hamiltonian originates from the 2nd term in the expansion of the expression

$$\sqrt{(mc^2)^2 + (c\vec{p}\,)^2} = mc^2 + \frac{\vec{p}^{\,2}}{2m} + \frac{(\vec{p}^{\,2})^2}{8m^3c^2} + \ldots .$$

Despite being the leading and largest term in the non-relativistic expansion, mc^2 is not included in the Schrödinger equation. Why not? How does E_0 change if one includes it?

(b) Calculate the first order correction $\Delta E_0^{(rel)}$ to the ground state energy due to the term $\frac{(\vec{p}^{\,2})^2}{8m^3c^2}$ in the non-relativistic expansion from part **(a)**. Express the final result in terms of E_0 and mc^2.

(c) If the nucleus has a finite radius $R > 0$ and its charge is assumed to be homogeneously distributed over its volume, then the $\frac{1}{r}$-dependence of the Coulomb potential is replaced by

$$\frac{1}{R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right)$$

for $r < R$ (and remains $\frac{1}{r}$ for $r > R$). Calculate the first order correction $\Delta E_0^{(R)}$ to the ground state energy E_0 due to the small parameter R/a. Express the final result in terms of E_0, R, a.

(d) Determine the correction $\Delta E_0^{(M)}$ if the nucleus has a finite mass M. Express the final result in terms of E_0, M, m

Hint: The formula

$$\int_0^\infty dx\, x^n e^{-x} = \Gamma(n+1)$$

with $\Gamma(n+1) = n!$ for integer n may be useful in some parts of this problem.

5. **(a)** Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two orthogonal states corresponding to two degenerate energy levels of a Hamiltonian H_0 each with energy E_0. Consider a constant Hermitian perturbation V that is capable of removing the degeneracy and spliting the levels into two levels that are ϵ apart.

(i) Show that it is always possible to define the states $|\psi_1\rangle$, $|\psi_2\rangle$ such that $\langle\psi_1|V|\psi_2\rangle$ is purely real. What are the values of the matrix elements $\langle\psi_1|V|\psi_2\rangle$ and $\langle\psi_2|V|\psi_1\rangle$.

Suppose that the system is initially in the state $|\psi_1\rangle$ and the perturbation V is introduced for a time T. $W(1 \to 2)$ is the probability of finding the system in state $|\psi_2\rangle$ at times $t > T$.

(ii) Show that $W(1 \to 2)$ is a periodic function of T with angular frequency $\epsilon/\hbar$.

(b) Consider the addition of two angular momentum operators according to $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}$. Eigenstates $|\ell_1, m_1\rangle$ are associated with the operators $\mathbf{L}_1^2$ and L_{1z}, eigenstates $|\ell_2, m_2\rangle$ are associated with the operators $\mathbf{L}_2^2$ and L_{2z}, and eigenstates $|L, M\rangle$ are associated with the operators $\mathbf{L}^2$ and L_z.

(i) For given finite ℓ_1 and ℓ_2 how many independent eigenstates of the form $|\ell_1, m_1\rangle|\ell_2, m_2\rangle$ does the system possess.

(ii) In terms of the quantum numbers (ℓ_1, m_1) and (ℓ_2, m_2) determine the values that are allowed for the quantum numbers (L, M).