Preliminary Exam: Quantum Mechanics, Friday August 25, 2017. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. Use the blue solution books and put the solution to each problem in a separate blue book and put the number of the problem on the front of each blue book. Be sure to put your name on each blue book that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla\psi = \mathbf{e}_x\frac{\partial\psi}{\partial x} + \mathbf{e}_y\frac{\partial\psi}{\partial y} + \mathbf{e}_z\frac{\partial\psi}{\partial z} = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} = \mathbf{e}_\rho\frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z}.$$

Hermite polynomial $= H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$, $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2-2$

Laguerre $= L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right)$, associated Laguerre $= L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r)$.

Legendre polynomial $= P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$,

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

associated Legendre polynomial $= P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$

spherical harmonic $= Y_l^m(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi}$,

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \ ,\ Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \ ,\ Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1) \ ,\ Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \ ,\ Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm2i\phi}$$

spherical Bessels : $j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right)$, $n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right)$,

with asymptotic behavior $j_\ell(r) \to \frac{\cos(r-\ell\pi/2-\pi/2)}{r}$, $n_\ell(r) \to \frac{\sin(r-\ell\pi/2-\pi/2)}{r}$.

$$j_0(r) = \frac{\sin r}{r} \ ,\ n_0(r) = -\frac{\cos r}{r} \ ,\ j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \ ,\ n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \ ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \ ,\ n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \ .$$

1. Consider the nodal behavior of solutions to the nonrelativistic Schrödinger equation and answer the following:

   (**a**) For a particle moving in a one-dimensional potential, assume there are two eigenfunctions ψ_1 and ψ_2 with corresponding eigenvalues $E_1 < E_2$. Show that between any two consecutive zeros (nodes) of ψ_1, there exists at least one node of ψ_2.

   Hint: Consider the Wronskian

   $$W[\psi_1, \psi_2] = \psi_1 \psi_2' - \psi_1' \psi_2,$$

   and find an expression for the derivative W' in terms of E_1, E_2, ψ_1 and ψ_2. Choose an interval of two consecutive zeros of ψ_1, say $[a, b]$ where $b > a$. Show that $W(b) - W(a)$ has the opposite sign to that of the integral of W' over the interval $[a, b]$, if ψ_2 does not change sign in this interval.

   (You may asssume that the potential is bounded in this interval $[a, b]$.)

   (**b**) When the radial solutions to a three-dimensional Hydrogen atom are considered:

   (**i**) is the result in part (**a**) valid for $2s$ and $3s$ wavefunctions. Explain your anwer using a sketch.

   (**ii**) is the result in part (**a**) valid for $2s$ and $3d$ wavefunctions. Explain your anwer using a sketch.

   (**iii**) In case (**ii**) which one of the $2s$ and $3d$ wavefunctions functions extends furthest?

2. Consider the following Hamiltonian for two spin 1/2 particles (A) and (B):

   $$H = \lambda \boldsymbol{\sigma}(A) \cdot \boldsymbol{\sigma}(B)$$

   where $\boldsymbol{\sigma}(A) = (\sigma_1, \sigma_2, \sigma_3)$, $\boldsymbol{\sigma}(B) = (\sigma_1, \sigma_2, \sigma_3)$ are defined using the Pauli spin matrices and λ is a constant.

   (**a**) Express H in terms of raising and lowering operators for the individual spins and the projections of the individual spins along the z-axis.

   (**b**) Write down a 4×4 matrix representation with respect to the basis set

   $$\{|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle\}$$

   defined using spin projections along the z-axis.

   (**c**) Find the eigenvalues and eigenvectors of the Hamiltonian in this basis.

3. The time-independent Schrödinger equation for a nonrelativistic particle moving in an N-dimensional space (N =1, 2, or 3) can be written in the coordinate representation:

$$\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}), \qquad \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(r),$$

where $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$ is the momentum operator, m is the particle mass, and $V(r)$ is a scalar potential.

(**a**) Derive the Schrödinger equation for the wave function $a(\mathbf{p})$ in the momentum representation:

$$a(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{N/2}} \int d^N r \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{r}}{\hbar}\right) \psi(\mathbf{r}).$$

(**b**) Write down this equation for the one-dimensional ($N = 1$) attractive potential described by the Dirac delta function $V(r) = -V_0\delta(r/b)$, where V_0 and b are positive constants. Starting from the Schrödinger equation in momentum space, derive the energy eigenvalue of the single bound state supported by this potential.

(**c**) What is the normalized momentum-space wave function $a(p)$ corresponding to the eigenvalue you found in part (**b**)?

4. A beam of mono-energetic particles each with energy E and mass m is scattered by a general spherically symmetric potential $U(r)$ that vanishes as $r \to \infty$. The scattering amplitude $f(\theta)$ can be expressed via the partial wave scattering amplitudes $f_\ell(\theta)$ and the scattering phase shifts $\delta_\ell(k)$ as

$$f(\theta) = \sum_{\ell=0}^{\ell=\infty} f_\ell(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\ell=\infty} (2\ell+1)\left[e^{2i\delta_\ell(k)} - 1\right] P_\ell(\cos\theta),$$

where ℓ is the angular momentum, θ is the scattering angle, $P_\ell(\cos\theta)$ are Legendre polynomials, and $k = (2mE/\hbar^2)^{1/2}$ is the wave number.

(**a**) Show that the total scattering cross section σ can be calculated using the imaginary part of the forward scattering amplitude (the optical theorem) as:

$$\sigma = \frac{4\pi}{k}\mathrm{Im}[f(\theta = 0)].$$

(**b**) Consider the potential well $U(r) = -U_0$ if $r < r_0$, and $U(r) = 0$ if $r > r_0$, where U_0 is a positive constant.

(**ii**) For this potential find the scattering phase shift $\delta_{\ell=0}(k)$ for s-wave scattering using the solution of the Schrödinger equation for the spherical wave with $\ell = 0$.

(**ii**) For this potential calculate the scattering cross section in the limit $k \to 0$ knowing that in this limit only s-wave scattering is important.

5. (**a**) A quantum-mechanical system has a time-independent Hamiltonian H_0 and an eigenspectrum of states $|n\rangle$ with energies E_n. While in its ground state it is subjected to a time-dependent perturbation $V(t)$ starting at a time $t = 0$. Derive the first order probability for finding this system in any other of its states at a later time t.

(**b**) Consider a rigid rotator (i.e. a bar shaped system of fixed separation) of moment of inertia I about an axis through its center perpendicular to the direction of the bar, with Hamiltonian $H_0 = \mathbf{L}^2/2I$ and electric dipole moment $\mathbf{d}$. Suppose that while it is in its ground state it is subjected to a perturbation

$$V(t) = -\mathbf{d} \cdot \mathbf{E}(t)$$

due to a time-dependent external electric field

$$\mathbf{E}(t) = \hat{\mathbf{z}} E_0 e^{-t/\tau},$$

which points in the z-direction and which is switched on at time $t = 0$. Here E_0 is a time-independent constant. Determine to which of its excited states the rotator can make transitions in lowest order in $V(t)$, and calculate the transition probabilities for finding the rotator in each of these states at time $t = \infty$.

(**c**) If instead of being in its ground state, the rotator was in a state with angular momentum $L = 3$ at the time $V(t)$ was switched on, determine the states to which it would be able to make transitions in lowest order in $V(t)$. (For this part do not calculate the transition probabilities.)