Preliminary Exam: Quantum Mechanics, Friday August 26, 2016. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla\psi = \mathbf{e}_x \frac{\partial \psi}{\partial x} + \mathbf{e}_y \frac{\partial \psi}{\partial y} + \mathbf{e}_z \frac{\partial \psi}{\partial z} = \mathbf{e}_r \frac{\partial \psi}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi} = \mathbf{e}_\rho \frac{\partial \psi}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial \psi}{\partial \phi} + \mathbf{e}_z \frac{\partial \psi}{\partial z}.$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \;,\; H_0(x)=1 \;,\; H_1(x) = 2x \;,\; H_2(x) = 4x^2-2$$

$$\text{Laguerre} = L_n(r) = e^r \frac{d^n}{dr^n}\left(r^n e^{-r}\right) \;,\; \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q \frac{d^q}{dr^q} L_{n+q}(r) \;.$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \;,\; P_0(x)=1 \;,\; P_1(x) = x \;,\; P_2(x) = \frac{1}{2}(3x^2-1) \;,$$

$$\int_{-1}^{+1} dw P_\ell(w) P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P^m_l(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y^m_l(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P^m_l(\cos\theta)e^{im\phi} \;,$$

$$Y^0_0 = \left(\frac{1}{4\pi}\right)^{1/2} \;,\; Y^0_1 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \;,\; Y^{\pm1}_1 = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y^0_2 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \;,\; Y^{\pm1}_2 = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \;,\; Y^{\pm2}_2 = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = (-1)^\ell r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\sin r}{r}\right) \;,\; n_l(r) = (-1)^{(\ell+1)} r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\cos r}{r}\right) \;,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} \;,\; n_\ell(r) \to \frac{\sin(r-\ell\pi/2-\pi/2)}{r} \;.$$

$$j_0(r) = \frac{\sin r}{r} \;,\; n_0(r) = -\frac{\cos r}{r} \;,\; j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \;,\; n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \;,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \;,\; n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \;.$$

1. A system with energy E_1 is an eigenstate $|1\rangle$ of a time-independent Hamiltonian H_0, when a perturbation $V(t)$ is switched on at time $t = 0$.

   (a) Derive the probability in first-order perturbation theory that the system will be found in a different eigenstate $|2\rangle$ with energy E_2 at a time $t \to \infty$.

   (b) Suppose now that the system is a single spinless particle of mass M and charge e in a central Coulomb potential and the quantum numbers of the initial state are labeled $|n, \ell, m\rangle$ in the usual way. The perturbation is a magnetic field $B(t) = B_0 e^{-\lambda t}$ pointing in the x direction, which adds a term $eL_x B(t)/(2Mc)$ to the Hamiltonian. Using the result of (a), find

   (i) the quantum number selection rules for allowed transitions to a second state $|n', \ell', m'\rangle$,

   (ii) the associated energy differences for the allowed transitions,

   (ii) the probabilities for each allowed transition.

2. (a) A rotation of a system of angular momentum $\mathbf{J}$ through an angle θ about an axis parallel to a vector $\hat{\mathbf{n}}$ of unit length is represented by the unitary operator $U = \exp(-i\theta\hat{\mathbf{n}} \cdot \mathbf{J}/\hbar)$. Show that in the case of a spin-half particle, this reduces to

$$U = \cos(\theta/2)I - i(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\sin(\theta/2)$$

   where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ represents the three Pauli matrices.

   (b) A beam of neutral spin one-half fermions with a net magnetic dipole moment enters a Stern-Gerlach apparatus traveling along the z axis. The Stern-Gerlach apparatus is oriented so that it measures the projection of the spin along axis $\hat{a}$, which lies in the xy plane. The beam then enters a second Stern-Gerlach apparatus which has been rotated about the z axis by angle θ so that it measures the spin projection along a different axis $\hat{b}$, also in the xy plane. Use the result of part (a) to show that the spin-up and spin-down states of the fermion with respect to the second magnet are related to those of the first magnet as

$$|\hat{\mathbf{b}} :\uparrow\rangle = \cos(\theta/2)|\hat{\mathbf{a}} :\uparrow\rangle - i\sin(\theta/2)|\hat{\mathbf{a}} :\downarrow\rangle$$

   and

$$|\hat{\mathbf{b}} :\downarrow\rangle = \cos(\theta/2)|\hat{\mathbf{a}} :\downarrow\rangle - i\sin(\theta/2)|\hat{\mathbf{a}} :\uparrow\rangle.$$

   Reminder: A Stern-Gerlach apparatus is a device for setting up an inhomogeneous magnetic field.

3. (a) Show that for a one-dimensional bound system with a potential $V(x)$ that is everywhere finite, the eigenvalues of the Schrödinger equation have to be nondegenerate. Give an example of a two- or three-dimensional system where some eigenvalues are degenerate.

   (b) Find the eigenstates and eigenvalues of the Schrödinger equation for the one-dimensional potential

$$V(x > 0) = Kx^2/2, \qquad V(x \leq 0) = \infty.$$

   (Hint: You may find it useful to express the eigenstates and eigenvalues in terms of the one-dimensional harmonic oscillator solutions.)

4. The Hamiltonian of a charged particle in a magnetic field is given by the formula:

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2,$$

where e and m are the particle charge and mass and $\mathbf{A}(\mathbf{r})$ is the vector potential of the magnetic field.

(a) Find the most general conditions under which the particle Hamiltonian can be written in the form:

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e}{mc}\mathbf{A}(\mathbf{r})\cdot\mathbf{p} + \frac{\mathbf{e^2}}{\mathbf{2mc^2}}\mathbf{A^2}(\mathbf{r})$$

(b) Show that the velocity obeys $m\mathbf{v} = \mathbf{p} - (e/c)\mathbf{A}$, calculate the commutation relations for velocity projections $[v_i, v_j]$, and express them via the components B_i of the magnetic field where $(i, j) = (x, y, z)$.

(c) Determine the energy and wave function of the ground state in a uniform magnetic field $\mathbf{B}$ of magnitude B, if the magnetic quantum number m is zero, and in cylindrical coordinates the vector potential is of the form

$$A_\phi = \frac{1}{2}B\rho, \qquad A_\rho = 0, \qquad A_z = 0.$$

5. A particle of the mass m and energy $E = \hbar^2 k^2/2m$ moves in an arbitrary potential $V(\mathbf{r})$.

(a) Show that the solution to the integral equation

$$\psi_k(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}' \frac{e^{ik\cdot|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} V(\mathbf{r}')\psi_k(\mathbf{r}')$$

satisfies the Schrödinger equation that describes the particle's motion.

(b) Show for $E = 0$ that an integral form of the Schrödinger equation for a spherically symmetric (s-wave) wave function and spherically symmetric potential $V(r)$ is given by the expression:

$$\psi_s(\mathbf{r}) = 1 - \frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}' V(r')\psi_s(r')g(r, r'),$$

where

$$g(r, r') = \theta(r - r')\frac{1}{r} + \theta(r' - r)\frac{1}{r'}.$$

(c) Derive an analytic expression for the s-wave scattering amplitude due to a spherically symmetric short-range potential (i.e. one for which $V(r > r_0) = 0$ for a finite r_0) that can be considered as a perturbation.

Hint: The asymptotic behavior of the wave function of the particle at large distances $r \gg r_0$ is $\psi(\mathbf{r}) = e^{ikr} + f_s/r$, where f_s is the s-wave scattering amplitude.