Preliminary Exam: Quantum Mechanics, Friday August 21, 2015. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Hermite polynomial $= H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$, $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2-2$

Laguerre $= L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right)$, associated Laguerre $= L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r)$.

Legendre polynomial $= P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$,

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

associated Legendre polynomial $= P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$

spherical harmonic $= Y_l^m(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$,

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} , \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta , \quad Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1) , \quad Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} , \quad Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

spherical Bessels : $j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right)$, $n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right)$,

with asymptotic behavior $j_\ell(r) \to \frac{\cos(r-\ell\pi/2-\pi/2)}{r}$, $n_\ell(r) \to \frac{\sin(r-\ell\pi/2-\pi/2)}{r}$.

$$j_0(r) = \frac{\sin r}{r} , \quad n_0(r) = -\frac{\cos r}{r} , \quad j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} , \quad n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} , \quad n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} .$$

$$\int_0^\infty \exp(-ax^2) = \left(\frac{\pi}{4a}\right)^{1/2} .$$

Problem 1

Consider a 3-state system described by the Hamiltonian

$$H = \lambda \left|Q\right\rangle \left\langle Q\right| + \mu \left|L\right\rangle \left\langle R\right| + \mu \left|R\right\rangle \left\langle L\right|$$

where $\left|Q\right\rangle$, $\left|L\right\rangle$ and $\left|R\right\rangle$ represent position eigenkets of an electron in three different orthonormal states. (λ and μ have units of energy.)

(a) Find the eigenstates and eigenvalues of the Hamiltonian H defined above.

(b) Write down the time evolved state $|\psi(t = T)\rangle$ produced by this Hamiltonian for an initial state that is given by $|\psi(t = 0)\rangle = |R\rangle$.

(c) For the state $|\psi(t = T)\rangle$ obtained in part **(b)** find the probabilities that the particle can be found in the states (i) $|Q\rangle$, (ii) $|L\rangle$ after time T.

Problem 2

Consider the state represented by the wave function

$$\psi(x, y, z) = N \exp(-\alpha r^2)(x + y)z, \quad r^2 = x^2 + y^2 + z^2,$$

where the parameter α is real and positive.

(a) Find the normalization constant of this state as a function of the parameter α.

(b) Calculate the expectation values of $\mathbf{L}$ and $L^2 = \mathbf{L} \cdot \mathbf{L}$ in this state.

(c) Calculate the variance of $\mathbf{L}$ and L^2 in this state. (For any operator A the variance is defined as $\langle A^2\rangle - \langle A\rangle^2$.)

(d) A general three-dimensional wave function can be expanded in angular momentum eigenstates as $\psi(x, y, z) = \sum_{\ell,m} a_{\ell,m}\langle x, y, z|\ell m\rangle$. Which ℓ and m are contained in the above $\psi(x, y, z)$?

Problem 3

(a) Consider a particle of mass m in the one-dimensional potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0, \\ 0 & \text{for } 0 \le x \le a, \\ V_0 & \text{for } x > a, \end{cases}$$

where V_0 and a are positive constants. Determine the minimum value V_{crit} of V_0 for which at least one bound state exists, and calculate the binding energy E_1 in this critical case as V_0 approaches V_{crit} from above.

(b) For $n = 1, 2, 3, \ldots$, how many bound states are there if $V_0 = n\, V_{\text{crit}} + \epsilon$, where ϵ is positive and infinitesimally small? Hint: you might find it helpful to solve this problem graphically.

Problem 4

(a) Consider the Hamilton operator H whose spectrum $\{E_n\}$ is discrete, bounded from below and non-degenerate, and whose eigenfunctions $\phi_n(x)$ span the Hilbert space $\mathcal{H}$. Prove that an upper bound for the ground state energy E_0 of H is given in terms of any normalized wave-function $\psi(x) \in \mathcal{H}$ through the inequality

$$E_0 \le \int \psi^*(x) H \psi(x)\, dx\,.$$

(b) Consider a particle of mass m bound in the one-dimensional potential

$$V(x) = \begin{cases} A\,x & \text{for } x \ge 0, \\ \infty & \text{for } x < 0, \end{cases}$$

where A is real and positive. For the trial wave function $\psi(\eta, x > 0) = x \exp(-\eta\, x)$, $\psi(\eta, x \le 0) = 0$, use the variational method to derive an upper bound for the ground state energy of a system with this potential.

Problem 5

(a) Write down the Dirac equation for a free relativistic spin one-half particle of mass m and momentum $\mathbf{p} = (p_x, p_y, p_z)$, and show that every solution to it obeys $E^2 = \mathbf{p}^2 c^2 + m^2 c^4$.

(b) Determine all the solutions to this Dirac equation if the particle has momentum p_z in the z-direction.

(c) Write down the Dirac equation for a relativistic spin one-half particle of mass m and momentum $\mathbf{p}$ moving in a central potential $V(r)$.

(d) Show that for the Dirac Hamiltonian with a central potential $V(r)$ the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is not a constant of the motion.

(e) Then construct an appropriate angular momentum which is in fact conserved.

Hint: For any vectors $\mathbf{A}$ and $\mathbf{B}$ the Pauli matrices obey:

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B}) + i\sigma \cdot (\mathbf{A} \times \mathbf{B}).$$

A convenient representation of the Dirac matrices is given as:

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

A second common representation of the Dirac matrices is $\gamma^0 = \beta$, $\gamma^1 = \beta\alpha_x$, $\gamma^2 = \beta\alpha_y$, $\gamma^3 = \beta\alpha_3$. These matrices obey $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ $(\mu = 0, 1, 2, 3)$, where $g^{\mu\nu}$ is a diagonal matrix with $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$.