QUANTUM MECHANICS

Preliminary Examination

Friday 08/23/2013

9:00am - 1:00pm in P-121

Answer a total of **FOUR** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on sheets of paper stapled together. Make sure you clearly indicate who you are and what is the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

On the last page you will find some potentially useful formulas.

Problem 1. A quantum particle is confined to the interval $[-a, a]$. It is described by the time-dependent wave function

$$\psi(x,t) = \frac{1}{\sqrt{2a}}\{\cos(\frac{\pi}{2a}x)\exp[-i(\frac{\hbar\pi^2}{8ma^2})t] - \sin(\frac{\pi}{a}x)\exp[-i(\frac{\hbar\pi^2}{2ma^2})t]\}.$$

(a) Show that $\psi(x,t)$ is properly normalized and find the associated probability current $J(x,t)$.

(b) Compute the probability

$$P_{left}(t) = \int_{-a}^{0} |\psi(x,t)|^2 dx$$

for finding the particle in the left half of the interval.

(c) Compare the probabilities $P_{left}(0)$ and $P_{left}(\tau)$ where $\tau = \frac{4ma^2}{\hbar\pi}$ and show that $P_{left}(0) > P_{left}(\tau)$.

(d) Show that $P_{left}(0) - P_{left}(\tau) = \int_0^\tau J(0,t)dt$ and interpret this result.

Problem 2. The angular momentum operators for states with a certain l value can be expressed in a matrix representation as

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The above matrix elements are defined with respect to an orthonormal basis set $\mathcal{B} = \{|\phi_1\rangle, \ |\phi_2\rangle, \ |\phi_3\rangle\}$. For example, the $(i,j)^{th}$ element of L_x is $\langle\phi_i| L_x |\phi_j\rangle$, with $\langle\phi_i|\phi_j\rangle = \delta_{ij}$ $(i, j = 1, 2, 3)$.

(a) Identify the l value and a basis $\mathcal{B}$ appropriate to this matrix representation.

(b) Compute the matrices L_+, L_- and L^2 in your basis.

(c) For the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_3\rangle)$, compute the expectation values $\langle\psi| L_x |\psi\rangle$, $\langle\psi| L_y |\psi\rangle$ and $\langle\psi| L_z |\psi\rangle$.

Problem 3. For the oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$$

let us define the annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}), \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p}).$$

(a) Prove that the commutation relation for these operators is

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

(b) Find the expression of the Hamiltonian operator $\hat{H}$ in terms of these operators.

(c) Prove that the eigenstate of the operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

has the form

$$\langle x|\alpha\rangle = C \exp\left(-\frac{1}{2}\alpha^2 - \frac{1}{2}\frac{m\omega}{\hbar}x^2 + \sqrt{\frac{2m\omega}{\hbar}}\,\alpha x\right).$$

Find the normalization constant C so that the states are normalized as

$$\langle\alpha|\alpha\rangle = \exp\alpha\alpha^\star$$

(this is a bit unusual normalization, but very convenient).

(d) The vacuum state of the system corresponds to the zero eigenstate of the annihilation operator $\hat{a}|0\rangle = 0$, while the excited states are obtained by the action of the creation operator $(\hat{a}^\dagger)^n|0\rangle$ (up to normalization). Using the results of the previous questions, obtain the coordinate representation of the wavefunction of the first excited state of the oscillator.

Problem 4. The unperturbed Hamiltonian of a two-state system is given by

$$H_0 = E_1^0|1\rangle\langle 1| + E_2^0|2\rangle\langle 2|$$

The system is subjected to a time-dependent perturbation,

$$V(t) = \lambda\cos\omega t|1\rangle\langle 2| + \lambda\cos\omega t|2\rangle\langle 1|$$

where λ is real.

(a) At $t = 0$ the system is in the first eigenstate $|1\rangle$. Using time-dependent perturbation theory, and assuming $E_1^0 - E_2^0$ is not close to $\pm\hbar\omega$, find the probability that the system is in state $|2\rangle$ at time t.

(b) Why is this procedure not valid when $E_1^0 - E_2^0 \sim \pm\hbar\omega$?

Problem 5. Consider two particles, each with spin 1/2. Using an explicit matrix representation of the system in the uncoupled basis (S_1, S_2 diagonal), find the eigenstates and eigenvalues of total spin, $S = S_1 + S_2$. What are the Clebsch-Gordan coefficients relating the original basis to the one where S^2 is diagonal?

$$\ln N! \approx N \ln N - N \quad \text{as} \quad N \to \infty$$

$$\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad \text{with} \quad \text{Re}(\alpha) > 0$$

$$\int_0^{\infty} dx \; x \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + i \frac{p}{m\omega}\right)$$