Preliminary Exam: Quantum Mechanics Friday 8/26/2011, 9:00-13:00

Answer a total of **FOUR** questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad , \quad H_0(x) = 1 \quad , \quad H_1(x) = 2x \quad , \quad H_2(x) = 4x^2 - 2$$

$$\text{Laguerre} = L_n(r) = e^r \frac{d^n}{dr^n}\left(r^n e^{-r}\right) \quad , \quad \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q \frac{d^q}{dr^q} L_{n+q}(r)$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad , \quad P_0(x) = 1 \quad , \quad P_1(x) = x \quad , \quad P_2(x) = \frac{1}{2}(3x^2-1) \quad ,$$

$$\int_{-1}^{+1} dw P_\ell(w) P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P^m_l(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}} P_l(x)$$

$$\text{spherical harmonic} = Y^m_l(\theta,\, \phi) = (-1)^m \left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P^m_l(\cos\theta) e^{im\phi} \quad ,$$

$$Y^0_0 = \left(\frac{1}{4\pi}\right)^{1/2} \quad , \quad Y^0_1 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \quad , \quad Y^{\pm 1}_1 = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y^0_2 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \quad , \quad Y^{\pm 1}_2 = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \quad , \quad Y^{\pm 2}_2 = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels :} \quad j_l(r) = (-1)^\ell r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\sin r}{r}\right) \quad , \quad n_l(r) = (-1)^{(\ell+1)} r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\cos r}{r}\right) \quad ,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} \quad , \quad n_\ell(r) \to \frac{\sin(r - \ell\pi/2 - \pi/2)}{r} \quad .$$

$$j_0(r) = \frac{\sin r}{r} \quad , \quad n_0(r) = -\frac{\cos r}{r} \quad , \quad j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \quad , \quad n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \quad ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \quad , \quad n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \quad .$$

Q.1 Consider an electron moving in the (xy) plane in a constant, vertical external magnetic field. The Schrödinger equation takes the form

$$-\frac{\hbar^2}{2m}\left[\partial_x^2 + \left(\partial_y - i\frac{eBx}{\hbar})\right)^2\right]\psi(x,y) = E\psi(x,y)\,.$$

(a) Find the spectrum of this Hamiltonian using the ansatz $\psi_n = e^{iky}f_n(x)$: Show that f_n is, up to a normalization constant, a shifted eigenfunction of the usual harmonic oscillator ϕ_n, $f_n(x) = \phi_n(x - \hbar k/eB)$, and that the energies of the states are $E_n = (\hbar eB/m)(n+1/2)$ with $n = 0, 1, \ldots$. The states with a fixed n make a "Landau level."

For a large enough system the results should not depend on boundary conditions as long as the boundary conditions are such that the momentum operator $-i\hbar\partial_x$ is hermitian. Let us therefore study the electron in a rectangle with the sides L_x and L_y using periodic boundary conditions: $\psi(x,y) = \psi(x+L_x, y) = \psi(x, y+L_y)$ for any x and y.

(b) Show that the periodicity in y requires that only a quantized subset of values of k is allowed. Find these values.

(c) Now estimate the degeneracy of the Landau level n by counting the values of k such that each wave function $f_n(x)$ fits in the box in the x-direction. Thus show in the limit of large L_x and L_y [which permits one to neglect small boundary effects associated with nonperiodicity of the functions $f_n(x)$] that the degeneracy of a Landau level is $N = (eB/2\pi\hbar)L_xL_y$.

Q.2 A particle of mass m is traveling along the z axis with the wave number k. It scatters off a spherically symmetric potential $V(r)$ that effectively vanishes for $r > a$. After scattering the particle emerges with an asymptotic outgoing wave function:

$$\psi(r) = \left[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right]$$

at $r \gg a$, where $f(\theta)$ is the scattering amplitude. In the asymptotic region the exact solution to the Schrödinger equation may be also written as a sum of partial waves

$$\psi(r) = \sum a_\ell\left[j_\ell(kr)\cos\delta_\ell - n_\ell(kr)\sin\delta_\ell\right]P_\ell(\cos\theta),$$

where each a_ℓ is an appropriate expansion coefficient and δ_ℓ is a phase shift.

(a) In terms of the phase shifts, show that the scattering amplitude $f(\theta)$ is given by

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\,P_\ell(\cos\theta).$$

You might find the following identity useful for this calculation:

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell\,j_\ell(kr)\,P_\ell(\cos\theta).$$

(b) Consider the special case of the finite-depth square well potential $V(r > a) = 0$, $V(r < a) = -V_0$, where $V_0 > 0$ is a constant. Determine the s-wave phase shift δ_0 in the low-momentum limit with $ka \ll 1$.

Q.3 Consider a one-dimensional harmonic oscillator with mass m and frequency ω in an energy eigenstate $|n\rangle$. Very slowly, a force acting on the oscillator is turned on from zero to a "small" value F.

(a) Into what state does the initial state $|n\rangle$ evolve?

(b) How much does the expectation value of the position change?

It may be helpful to recall the expression for the lowering operator of the harmonic oscillator: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)$.

Q.4 The Heisenberg equation of motion for an operator O in quantum mechanics reads

$$i\frac{d}{dt}O = [O, H]\,.$$

Consider the Hamiltonian of an anharmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{\lambda}{4!}x^4\,.$$

Here and below the system of units is such that $\hbar \equiv 1$.

(a) Write down explicitly Heisenberg equations of motion for the operators x, p, x^2, and p^2.

Taking the expectation value of the Heisenberg equation in a state $|\Psi\rangle$ we obtain the equation of motion for the average of O,

$$i\frac{d}{dt}\langle\Psi|O|\Psi\rangle = \langle\Psi|[O,H]|\Psi\rangle\,.$$

One can think either of the operator O (Heisenberg picture) or the wave function Ψ (Schrödinger picture) as time dependent. In general this equation is not closed because its right-hand side involves expectation values of operators distinct from O. Let us however assume that the Schrödinger picture wave function Ψ can at all times be approximated by a Gaussian

$$\Psi(x) = \left(\frac{1}{\pi G}\right)^{\frac{1}{4}} \exp\{-\frac{1}{2}(x - X)(G^{-1} + i\Sigma)(x - X) + ixP\}\,.$$

Here the real parameters X, P, G, and Σ depend on time, and the state is normalized to one. In this "Gaussian approximation" the expectation values of the Heisenberg equations you have derived in (**a**) close, and reduce to a set of equations of motion for the parameters of the Gaussian.

(b) Calculate the expectation values $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle xp\rangle$ in terms of the parameters X, P, G, and Σ.

(c) Let us now additionally set $X = 0$ and $P = 0$. Derive the equations of motion for G and Σ (known as "squeezing parameters") in this approximation. To derive these equations you can use the following relation between the expectation values in a Gaussian state

$$< px^3 >= 3 < px >< x^2 > \,.$$

Also remember $x^3 p = (px^3)^\dagger$.

Q.5 Consider the addition of two angular momentum operators according to $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}$. Eigenstates $|\ell_1, m_1\rangle$ are associated with the operators $\mathbf{L}_1^2$ and L_{1z}, eigenstates $|\ell_2, m_2\rangle$ are associated with the operators $\mathbf{L}_2^2$ and L_{2z}, and eigenstates $|L, M\rangle$ are associated with the operators $\mathbf{L}^2$ and L_z.

(a) In terms of the quantum numbers (ℓ_1, m_1) and (ℓ_2, m_2), determine (i.e. derive as well as state the answer) the values allowed for the quantum numbers (L, M).

(b) In terms of the basis vectors $|\ell_1, m_1\rangle|\ell_2, m_2\rangle$, construct the particular eigenstates $|L, M\rangle$ that possess the two highest allowed positive M values.