Preliminary Exam: Quantum Physics 8/27/2010, 9:00-3:00

Answer a total of **SIX** questions of which at least **TWO** are from section **A**, and at least **THREE** are from section **B**. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int_0^\infty dx\ x^n e^{-ax} = \frac{n!}{a^{n+1}}, \qquad \int_0^\infty dx\ e^{-a^2x^2} = \frac{\pi^{1/2}}{2a}, \qquad \int_0^\infty dx\ xe^{-a^2x^2} = \frac{1}{2a^2},$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \;,\; H_0(x) = 1 \;,\; H_1(x) = 2x \;,\; H_2(x) = 4x^2 - 2$$

$$\text{Laguerre} = L_n(r) = e^r \frac{d^n}{dr^n}\left(r^n e^{-r}\right) \;,\; \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q \frac{d^q}{dr^q} L_{n+q}(r) \;.$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \;,\; P_0(x) = 1 \;,\; P_1(x) = x \;,\; P_2(x) = \frac{1}{2}(3x^2-1) \;,$$

$$\int_{-1}^{+1} dw P_\ell(w) P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} \;,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \;,\; Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \;,\; Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \;,\; Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \;,\; Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \;,\; n_l(r) = (-1)^{(\ell+1)} r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \;,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} \;,\; n_\ell(r) \to \frac{\sin(r - \ell\pi/2 - \pi/2)}{r} \;.$$

$$j_0(r) = \frac{\sin r}{r} \;,\; n_0(r) = -\frac{\cos r}{r} \;,\; j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \;,\; n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \;,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \;,\; n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \;.$$

A convenient representation of the Dirac matrices is given as:

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \ \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Section A: Statistical Mechanics

A.1 A vessel with volume V_1 contains N molecules of an ideal gas held at temperature T and pressure P_1. The energy of a molecule in the gas may be written in the form

$$E_k(p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \epsilon_k$$

where ϵ_k denotes the energy levels associated with the internal states of the gas molecules.

(a) Calculate the partition function and the Helmholtz free energy of the ideal gas.

(b) Calculate the entropy of the ideal gas and express it in terms of the gas pressure.

(c) Now consider another vessel of volume V_2, also at the temperature T, containing the same number of molecules N, but held at pressure P_2. The vessels are connected to permit the gases to mix without doing any work. Calculate the resulting entropy change of the system. Check whether your answer makes sense by considering the limit in which $V_1 = V_2$ and $P_1 = P_2$, i.e whether or not your answer correctly yields the entropy of mixing of two identical systems having the same volume and pressure.

A.2 Consider a system of $N = 10^{22}$ electrons in a box of volume $V = 1000\ \text{cm}^3$. The walls of the box are infinitely high potential barriers. Calculate the following and show the dependence on the relevant physical parameters:

(a) The density of states of the electron gas.

(b) The average kinetic energy at temperature $T = 0$.

(c) The pressure on the walls at $T = 0$.

(d) The heat capacity at $T \ll T_F$ where T_F is the Fermi temperature of the electron gas.

A.3 A one-dimensional quantum harmonic oscillator with energy levels $\epsilon_n = \hbar\omega(n + 1/2)$ is in thermal equilibrium with a thermal bath at temperature T.

(a) What is a partition function of this oscillator?

(b) What is the average oscillator energy $< \epsilon >$ as a function of T in the two limits $T \to 0$ and $T \to \infty$?

(c) Calculate the energy fluctuation $< \epsilon^2 > - < \epsilon >^2$ for the oscillator at temperature T.

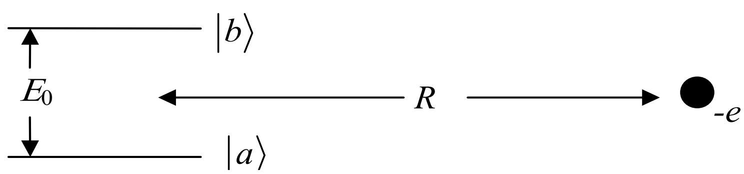
Section B: Quantum Mechanics

B.1 A one-electron, two-level atom with eigenstates $|a\rangle$ and $|b\rangle$ is located at a distance R from an ion with charge $-e$. The atom is perturbed by the electric field of the ion, with an interaction Hamiltonian that can be expressed in the electric dipole approximation as

$$H' = e\mathbf{E}(R) \cdot \mathbf{r}.$$

Here $\mathbf{E}(R)$ is the electric field produced by the ion at the position of the atom and $\mathbf{r}$ is the coordinate of the atomic electron. Because of the assumption of a pure two-level atom, the interaction is essentially one-dimensional, $e\mathbf{E}(R) \cdot \mathbf{r} \to eE(R)r$. The unperturbed energy level separation is E_0, and the matrix element of er between the two states has value M:

$$\langle b|er|a\rangle = M.$$



(a) If the states $|a\rangle$ and $|b\rangle$ have opposite parity, show that there is no first-order shift in the energy of the ground state $|a\rangle$.

(b) If the atom is initially in state $|a\rangle$, show that the lowest-order interaction energy with the ion can be written in the form

$$\Delta E = -\frac{C_n}{R^n}.$$

Find the numerical value of the exponent n, and derive an explicit expression for C_n. Determine the force on the atom associated with this interaction.

(c) Because of the slight mixing of states $|a\rangle$ and $|b\rangle$ by the ionic field, there is a small induced dipole moment in the atom. Find its value. Show that the classical force on this dipole is in reasonable agreement with the force that you calculated in part **(b)**.

B.2 An atomic trampoline is formed by an atom of mass m that bounces up and down in the vertical z direction from a perfectly reflecting horizontal surface at $z = 0$, in the presence of a gravitational potential mgz. Assume that the atom is structureless and that the reflection is perfectly elastic.

(a) Use an appropriate semiclassical approximation of your choosing (Bohr-Sommerfeld quantization, the WKB approximation, etc.) to find the approximate energy of the quantum ground state of this system.

(b) The wave function of a falling plane wave that has zero momentum at t=0 can be written as

$$\psi(z,t) = e^{-i\gamma tz}e^{i\Phi(t)}, \quad \gamma \equiv \frac{mg}{h}, \quad \Phi(t) \equiv -\frac{\hbar\gamma^2 t^3}{6m}.$$

Show by direct substitution that this is a solution to the Schrödinger equation. Estimate the period T of the ground-state of the atomic trampoline by placing an appropriate constraint on the phase change from $t = 0$ to $t = T$ as measured at a fixed position z.

(c) By using appropriate semiclassical or classical approximations to relate the energy to the period, show that the results you obtained in parts (**a**) and (**b**) are consistent to within no more than a small numerical factor.

B.3 A system occupies an eigenstate $|1\rangle$ of its time-independent Hamiltonian H_0, when a perturbation $V(t)$ is switched on at time $t = 0$.

(a) Derive the probability in first-order perturbation theory that the system will be found in a different eigenstate $|2\rangle$ at a later time t.

(b) Suppose now that the system is a single spinless particle of mass M and charge e in a central potential and the initial state is labeled $|n, \ell, m\rangle$ in the usual way. The perturbation is a magnetic field $B(t) = B_0 e^{-\lambda t}$ in the x direction, which adds a term $eL_x B(t)/(2Mc)$ to the Hamiltonian. Using the result of (**a**), find the selection rules for a transition to a second state $|n', \ell', m'\rangle$ and determine the probability for the allowed transitions.

B.4 Consider the quantum-mechanical creation and annihilation operators $a^\dagger$ and a, which satisfy the fundamental commutation relation

$$[a, a^\dagger] = \hbar I,$$

where I is the unit matrix. For such states introduce a vacuum state $|\Omega\rangle$, which obeys $a|\Omega\rangle = 0$.

(a) Show that the coherent state

$$|\alpha\rangle = e^{-\hbar|\alpha|^2/4} e^{\alpha a^\dagger} |\Omega\rangle$$

is an eigenstate of the annihilation operator α, and determine the associated eigenvalue.

(b) Expand the coherent state $|\alpha\rangle$ in terms of normalized eigenstates of the number operator $N = a^\dagger a$, and hence determine the probability for the coherent state $|\alpha\rangle$ to contain n quanta.

(c) Show that the coherent state $|\alpha\rangle$ is a "minimum uncertainty state", in the sense that it saturates the uncertainty principle bound for position and momentum operators, where $x = (a^\dagger + a)/2$, $p = i(a^\dagger - a)/2$.

B.5 A three-dimensional spherically symmetric square-well potential $V(r)$ has the form

$$V(r \leq R) = -V_0, \qquad V(r > R) = 0,$$

where V_0 is a finite positive constant.

(a) Derive an inequality that V_0 and R must satisfy so that the potential supports at least one bound state.

(b) Derive an expression for the total scattering cross section on this potential when the kinetic energy of an incident beam is so low that only s-wave scattering is significant.

B.6

(a) Write down the Dirac equation for a free relativistic spin one-half particle of mass m, and obtain the wave function which describes an electron moving in the z direction with spin up and negative energy.

(b) Consider the vector potential $\vec{A} = (-yB, 0, 0)$ where B is a pure space and time independent constant. What magnetic field does $\vec{A}$ represent?

(c) An electron is placed in this magnetic field. Write down the Dirac equation for this system.

(d) Make the non-relativistic reduction of this Dirac equation to obtain the Schrödinger equation and its first relativistic correction. In particular determine the g factor of the electron associated with the coupling of its spin to the magnetic field.