Preliminary Exam: Quantum Physics 8/25/2006, 9:00-3:00

Answer a total of **SIX** questions of which at least **TWO** are from section **A**, and at least **THREE** are from section B. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and each sheet of paper you submit.

Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int_0^\infty dx\, e^{-a^2x^2} = \frac{\pi^{1/2}}{2a}, \quad \int_0^\infty dx\, x e^{-a^2x^2} = \frac{1}{2a^2},$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \;,\; H_0(x) = 1 \;,\; H_1(x) = 2x \;,\; H_2(x) = 4x^2 - 2$$

$$\text{Laguerre} = L_n(r) = e^r \frac{d^n}{dr^n}\left(r^n e^{-r}\right) \;,\; \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q \frac{d^q}{dr^q} L_{n+q}(r) \;.$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \;,\; P_0(x) = 1 \;,\; P_1(x) = x \;,\; P_2(x) = \frac{1}{2}(3x^2-1) \;,$$

$$\int_{-1}^{+1} dw P_\ell(w) P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} \;,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \;,\; Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \;,\; Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \;,\; Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \;,\; Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \;,\; n_l(r) = (-1)^{(\ell+1)} r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \;,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} \;,\; n_\ell(r) \to \frac{\sin(r-\ell\pi/2-\pi/2)}{r} \;.$$

$$j_0(r) = \frac{\sin r}{r} \;,\; n_0(r) = -\frac{\cos r}{r} \;,\; j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \;,\; n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \;,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \;,\; n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \;,$$

A convenient representation of the Dirac matrices is given as:

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ,\; \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} ,\; \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ,\; \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Section A: Statistical Mechanics

A.1 Consider a system of two noninteracting identical particles that may occupy any of three energy levels: $e_n = ne,\ n = 0, 1, 2$. The energy state with energy $e_1 = e$ is doubly degenerate. The system is in thermal equilibrium at temperature T. For each of the following three cases carefully enumerate the energy eigenstates of the system, and then calculate the canonical partition function and the average energy in each of the cases.

(a) Particles obey Fermi statistics

(b) Particles obey Bose statistics

(c) Particles are distinguishable and obey Boltzmann statistics

A.2 (a) Consider a degenerate, spin one-half, non-interacting Fermi gas at zero temperature $T = 0$. Find an expression for the energy when the gas contains N particles in a volume V.

(b) Given such an expression for the internal energy of a Fermi system at zero temperature, how does one determine the pressure?

Hint: Use the thermodynamic relation:

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p \ .$$

(c) Find the isothermal compressibility

$$K = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

of this gas at zero temperature $T = 0$.

It may be of help to recall that the Helmholtz free energy $F = E - TS$ obeys the relation $p = -\left(\frac{\partial F}{\partial V}\right)_T$.

A.3 Consider electromagnetic radiation that fills a cavity of volume V with a Planck energy distribution. Initially ω_i is the frequency of the maximum of the curve of $u_i(\omega)$, the energy density per unit angular frequency versus ω. If the volume is expanded adiabatically to $2V$, what is the final peak frequency ω_f of the $u_f(\omega)$ distribution curve?

Section B: Quantum Mechanics

B.1 (a) Write down the Dirac equation for a relativistic spin 1/2 particle of mass m moving in a central potential $V(r)$.

(b) Identify the Hamiltonian for this system and show that the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is not a constant of the motion.

(c) Then construct an appropriate angular momentum which is in fact conserved.

B.2 The Hamiltonian responsible for parity violation in a one-electron atom can be parameterized as $H_{\rm PNC} = A\,\mathbf{s}\cdot\left(\mathbf{p}\delta^3(\mathbf{r}) + \delta^3(\mathbf{r})\mathbf{p}\right)$, where A is a very small real constant, $\mathbf{s}$ is the electron spin, and $\mathbf{p}$ is its momentum. Consider a simplified model of an alkali atom, treating it as a two-level, single-electron system with levels $|ns_{1/2}\rangle$ and $|np_{1/2}\rangle$ (for example, $6s_{1/2}$ and $6p_{1/2}$ for the case of cesium).

(a) Use lowest-order perturbation theory to find an expression for the admixture γ of the $|np_{1/2}\rangle$ wave function into the $|ns_{1/2}\rangle$ state, in terms of a matrix element of $H_{\rm PNC}$. That is, find an expression for the parameter γ which appears in the first-order wave function, $|ns_{1/2}\rangle_{\rm mod} = |ns_{1/2}\rangle + \gamma\,|np_{1/2}\rangle$.

(b) Now evaluate the perturbation matrix element $\langle np_{1/2}|\,H_{\rm PNC}\,|ns_{1/2}\rangle$ explicitly, by taking advantage of the fact that only the electronic wave functions near $r = 0$ are relevant due to the Dirac delta functions in $H_{\rm PNC}$. In this small-r region the radial parts of the wave functions can be accurately approximated with hydrogen-like wave functions, to within a Z-independent pair of multiplicative real constants b and c:

$$\psi_{n,s}(r) \simeq b\frac{\sqrt{Z}}{a_0^{3/2}}e^{-Zr/a_0} \quad , \quad \psi_{n,p}(r) \simeq c\left(\frac{Z}{a_0}\right)^{3/2}\frac{r}{a_0}e^{-Zr/a_0} .$$

To simplify the calculation, we will average over the angular dependence. After writing out $\mathbf{p} \propto -i\hbar\nabla$ and performing this average (you do not need to do this yourself!), the matrix element of $\mathbf{s}\cdot\mathbf{p}$ can be replaced by a scalar version in a decoupled angular momentum basis,

$$\langle np_{1/2}|\,\mathbf{s}\cdot\mathbf{p}\,|ns_{1/2}\rangle_{\rm average} \propto -i\hbar\sqrt{3}\,\langle s=\tfrac{1}{2}, m_s=\tfrac{1}{2}|\,s_z\,|s=\tfrac{1}{2}, m_s=\tfrac{1}{2}\rangle\,\langle np, m_l=0|\,\nabla_z\,|ns, m_l=0\rangle .$$

In evaluating the angular part of the matrix element, you will probably find it useful to note that

$$\nabla_z = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} .$$

What is the dependence of the perturbation on the atomic number Z? Can you see why experiments are typically performed on cesium, not on hydrogen or lithium?

(c) In the above you should have found that the wave function admixture parameter γ is purely imaginary. In any case, demonstrate that γ cannot be real by showing that if it were, there would be a first-order Stark shift of the perturbed energy level $|ns_{1/2}\rangle_{\rm mod}$ in an external dc electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. This would violate time-reversal symmetry, which is not a property of this particular Hamiltonian even though it does violate parity.

B.3 Consider the addition of two angular momentum operators according to $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}$. Eigenstates $|\ell_1, m_1\rangle$ are associated with the operators $\mathbf{L}_1^2$ and L_{1z}, eigenstates $|\ell_2, m_2\rangle$ are associated with the operators $\mathbf{L}_2^2$ and L_{2z}, and eigenstates $|L, M\rangle$ are associated with the operators $\mathbf{L}^2$ and L_z.

(a) In terms of the quantum numbers (ℓ_1, m_1) and (ℓ_2, m_2) determine (i.e. derive as well as state the answer) the values which are allowed for the quantum numbers (L, M).

(b) In terms of the basis vectors $|\ell_1, m_1\rangle$ and $|\ell_2, m_2\rangle$ construct the particular eigenstates $|L, M\rangle$ which possess the three highest allowed positive M values. (If you do not wish to solve this part for the general ℓ_1, ℓ_2, for partial credit you can instead solve for the explicit case with $\ell_1 = 2$, $\ell_2 = 2$.)

B.4 A two-level system with levels $|a\rangle$ and $|b\rangle$ separated by energy $\hbar\omega_0$ is subjected to a weak perturbation from an electric field $\mathbf{E} = E_0\ \theta(t)\ \hat{\mathbf{z}}$ that is abruptly turned on at $t = 0$. (Here $\theta(t)$ is the Heaviside step function $\theta(t > 0) = 1$, $\theta(t < 0) = 0$.) Even though the field is subsequently constant, the level populations are time-dependent because of the abrupt turn-on. Assume that there is a real, non-vanishing electric dipole matrix element $d = \langle b|\, ez\, |a\rangle$ coupling the levels, so that the perturbation Hamiltonian can be written as the operator

$$H' = dE_0\theta(t)|a\rangle\langle b| + dE_0\theta(t)|b\rangle\langle a| \ .$$

(a) Using lowest-order time-dependent perturbation theory or other methods of your choosing, find the time-dependent population $P_b(t)$ in the upper level as a function of d, E_0, and ω_0. What is its maximum value? How rapidly does the population oscillate with time?

(b) Compare your result with the value of $P_b(t)$ calculated for a time-independent weak dc field $\mathbf{E} = E_0\hat{\mathbf{z}}$ by use of time-independent perturbation theory, or other methods of your choosing. Does the comparison make sense?

B.5 The expectation value of an operator $\Omega(\mathbf{r}, \mathbf{p}, t)$ in a state $\psi(\mathbf{r}, t)$ is given as $<\Omega>= \int d^3r\psi^*(\mathbf{r}, t)\Omega\psi(\mathbf{r}, t)$.

(a) Show for any $\psi(\mathbf{r}, t)$ which obeys the Schrodinger equation associated with the Hamiltonian $H = \mathbf{p}^2/2m + V(\mathbf{r})$ with real $V(\mathbf{r})$ that Ehrenfest's theorem holds, viz.

$$\frac{d}{dt} < \mathbf{r} >= \frac{< \mathbf{p} >}{m} \quad , \quad \frac{d}{dt} < \mathbf{p} >=< \mathbf{F} >= - < \frac{dV}{d\mathbf{r}} > \ .$$

(b) In the presence of electromagnetism an appropriate Hamiltonian is

$$H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - e\phi$$

where $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ are vector and scalar electromagnetic potentials. Derive the appropriate Ehrenfest's theorem in this case to explicitly obtain the Lorentz force law.

B.6 A one-dimensional quantum-mechanical particle of mass m is placed in an attractive one-dimensional delta function potential $V(x) = -\hbar^2\lambda\delta(x)/m$ with positive λ. The particle and the potential are located in a box with perfectly reflecting walls at $x = a/2$ and $x = -a/2$ (i.e. at the walls of the box the potential is given by $V(a/2) = V(-a/2) = \infty$).

(a) Determine the condition on the parameters of the problem for which the system will possess exactly one bound state with negative energy eigenvalue E, and give its wave function.

(b) For the same system determine the energy eigenvalues and eigenvectors for states with positive E.

(c) How many negative energy bound states would the system possess in the event that the coefficient λ were negative?

Hints: The delta function $\delta(x)$ is an even function of x (think what this means for the energy eigenfunctions), which is given as the derivative $d\theta(x)/dx = \delta(x)$ where the Heaviside function $\theta(x)$ obeys $\theta(x > 0) = 1$, $\theta(x < 0) = 0$. In terms of the Heaviside function a general function $f(x)$ can be written as $f(x) = f_1(x)\theta(x) + f_2(x)\theta(-x)$, with the the function $|x|$ for instance being given by $|x| = x\theta(x) - x\theta(-x)$.