Preliminary Exam: Quantum Physics 8/27/2004, 9:00-3.00

Answer a total of **SIX** questions of which at least **TWO** are from section **1**, and at least **THREE** are from section **2**. Put each of your solutions in a separate answer book.

Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int_0^\infty dx\, e^{-a^2x^2} = \frac{\pi^{1/2}}{2a}, \quad \int_0^\infty dx\, xe^{-a^2x^2} = \frac{1}{2a^2},$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

$$\text{associated Laguerre} = L_{n+l}^{2l+1}(r) = \sum_{k=0}^{n-l-1}(-1)^{k+2l+1}\frac{[(n+l)!]^2 r^k}{(n-l-1-k)!(2l+1+k)!k!}$$

$$\text{Legendre polynomial} = P_l(w) = \frac{1}{2^l l!}\frac{d^l}{dw^l}(w^2-1)^l$$

$$\text{associated Legendre polynomial} = P_l^m(w) = (1-w^2)^{|m|/2}\frac{d^{|m|}}{dw^{|m|}}P_l(w)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = R_l(r)\frac{\sin r}{r} + S_l(r)\frac{\cos r}{r}, \qquad n_l(r) = R_l(r)\frac{\cos r}{r} - S_l(r)\frac{\sin r}{r},$$

$$\text{where} \quad R_l(r) + iS_l(r) = \sum_{s=0}^{l}\frac{i^{s-l}(l+s)!}{2^s s!(l-s)!}r^{-s},$$

$$\text{and with asymptotic behavior} \quad j_\ell(r) \to \frac{\sin(r-\ell\pi/2)}{r}, \quad n_\ell(r) \to \frac{\cos(r-\ell\pi/2)}{r}.$$

Section 1: Statistical Mechanics

1.1 Consider a macroscopic harmonic oscillator, consisting of a particle of mass m and a spring with a constant α, which is embedded in a viscous fluid of temperature T. The friction between the oscillator and the fluid gives rise to a friction force that depends linearly on the velocity of the particle, with a friction coefficient γ.

(a) Show that the equation of motion for this system is given by

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = \frac{F_s(t)}{m} \quad ,$$

where F_s is a random force created by the stochastic impact of fluid molecules on the surface of the mass m, and $\omega_0^2 = \alpha/m$ denotes the square of the fundamental frequency of the oscillator.

(b) What is the mean square displacement $\langle x^2 \rangle$ of the oscillator at temperature T? What is the mean square velocity $\langle \dot{x}^2 \rangle$?

(c) Explain why, in thermal equilibrium, the random force F_s and the velocity $\dot{x}$ must be correlated. What is the correlation $\langle F_s \dot{x} \rangle$? (Hint: What is the power of the random force on the particle? What is the power dissipated by the friction term?)

(d) What is the mean square stochastic force $\langle F_s^2 \rangle$?

1.2 Consider a system of N particles with classical Hamiltonian

$$H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + U(\mathbf{q}) \quad ,$$

where $U(\mathbf{q})$ is the potential energy of the system.

(a) Show that the chemical potential μ in this system can be decomposed into an ideal and an excess part, $\mu = \mu_{ex} + \mu_{id}$, where μ_{id} is the chemical potential of a classical ideal gas at the same temperature and density, and μ_{ex} depends only on the temperature T, the volume V, and the potential energy function $U(\mathbf{q})$.

(b) At regular intervals during a molecular dynamics simulation of this system, an additional "test" particle is placed in the system at random locations, and the (instantaneous) change ΔU_t in potential energy resulting from that addition is calculated. The test particle is then removed, and the molecular dynamics simulation resumed.

(c) Show that the excess chemical potential μ_{ex} is given by

$$\mu_{ex} = -kT \ln \langle \exp - \frac{\Delta U_t}{kT} \rangle_{V,T} \quad .$$

Hint: Use the identity

$$\mu = \left(\frac{\partial A}{\partial N} \right)_{V,T} = A(N+1, V, T) - A(N, V, T) \quad .$$

1.3 Consider a system of eight identical particles with four distinct motional energy levels. How many distinguishable microstates are there for this system, if the particles have spins

(a) $J = 1/2$, (b) $J = 3/2$, (c) $J = 5/2$, (d) $J = 7/2$.

Section 2: Quantum Mechanics

2.1 Consider the quantum mechanical creation and annihilation operators $a^\dagger$ and a, which satisfy the fundamental commutation relations $[a,\, a^\dagger] = \mathbf{1}$, and which can be expressed in terms of position and momentum operators as

$$a = \frac{1}{\sqrt{2\hbar}}\left(x\sqrt{m\omega} + \frac{i}{\sqrt{m\omega}}p\right)$$

(a) If $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$ is the normalized eigenstate of the number operator, show that

$$\langle m|\, a\, |n\rangle = \sqrt{n}\,\delta_{m,n-1}$$

and

$$\langle n|x^2|n\rangle = \left(n + \frac{1}{2}\right)\frac{\hbar}{m\omega} \quad .$$

(b) Now consider a slightly anisotropic three-dimensional harmonic oscillator, with $\omega_x = \omega_y = \omega$, and $\omega_z^2 = \omega^2 + \bar{\omega}^2$, where $\bar{\omega} \ll \omega$. A charged particle moves in the field of this oscillator potential and is at the same time exposed to a uniform magnetic field directed in the x direction. Assuming the Zeeman splitting to be comparable to the splitting produced by the anisotropy of the harmonic potential, but small compared to $\hbar\omega$, calculate to first order the energies of the components of the first excited state.

(c) Check your answer in part (b) against the limiting cases of no anisotropy or no magnetic field.

2.2 Consider a particle moving in a one-dimensional periodic potential of period a:

$$V(x + a) = V(x) \quad .$$

(a) State and give a proof of Bloch's theorem for the behavior of the wavefunction of a particle in such a periodic potential under a translation through one lattice spacing.

(b) Explain carefully the physical consequences of Bloch's theorem for the spectrum of particles in such a periodic potential.

(c) Consider a long periodic array of binding delta function wells, each of which represents an atom in a one-dimensional periodic crystal:

$$V(x) = -g\sum_{l=-N}^{N}\delta(x - l\,a) \quad .$$

Use Bloch's theorem to compute the *discriminant* $\cos(K\,a)$, where K is the "quasi-momentum".

(d) Make a rough sketch of the spectrum for the potential in part (c) for two different values of the binding strength:

(i) when $\frac{mga}{\hbar^2} = 1$, **(ii)** when $\frac{mga}{\hbar^2} = 10$.

2.3 A particle of mass m is traveling along the z-axis with momentum of magnitude k. It scatters off a spherically symmetric potential $V(r)$ which vanishes for $r > a$. After scattering the particle emerges with an outgoing wave function

$$\psi(\bar{r}) = [e^{ikz} + f(\theta)e^{ikr}/r]$$

at $r >> a$ where $f(\theta)$ is the scattering amplitude. In the region $r >> a$ the exact solution to the Schrodinger equation may be also written as a sum of partial waves

$$\psi(\bar{r}) = \sum a_\ell [j_\ell(kr) cos\delta_\ell - n_\ell(kr) sin\delta_\ell] P_\ell(cos\theta)$$

where each a_ℓ is an appropriate normalization constant and each δ_ℓ is a phase shift.

(a) Compare these two expressions for the outgoing wave function to show that

$$f(\theta) = (1/k) \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} sin\delta_\ell P_\ell(cos\theta)$$

(b) Consider the case where the potential is taken to be a hard sphere of radius a

$$V(r) = \infty \quad , \quad r < a \; ; \; V(r) = 0 \quad , \quad r > a$$

(c) Find the total cross section in the limit where $a \ll \frac{1}{k}$.

(d) Consider the case $a \gg \frac{1}{k}$. Show that in the forward direction the various partial wave contributions to the scattering amplitude $f(\theta)$ add up coherently to produce a diffraction pattern of Fraunhofer type.

You may find the following formulas useful.

$$P_n(\cos\theta) \approx J_0(n\,\theta) \quad , \quad \theta \text{ small, } n \gg 1$$

$$\frac{d}{dz}\left[z^{n+1} J_{n+1}(z)\right] = z^{n+1} J_n(z)$$

2.4

(a) A quantum-mechanical system has a time-independent Hamiltonian H_0 and an eigenspectrum of states $|n\rangle$ with energies E_n. While in its ground state it is subjected to a time-dependent perturbation $V(t)$ starting at a time $t = 0$. Derive the first order probability for finding this system in any other of its states at a later time t.

(b) A one-dimensional harmonic oscillator is initially in its second excited state $| 2 >$. It is subjected to a perturbation

$$V(t \geq 0) = \alpha x^2 \exp(-t/\tau), \qquad x = \left(\frac{\hbar}{2m\omega}\right)^{1/2} (a + a^\dagger)$$

where τ is positive.

(i) To what states can it make transitions in first order perturbation theory?

(ii) Calculate the corresponding transition probabilities to these states after the perturbation has been applied for a long time ($t \to \infty$).

2.5 A one-dimensional wave packet is formed at time $t = 0$ by a Gaussian superposition of free particle plane waves

$$\psi(x,0) = \left(\frac{a^2}{2\pi^3}\right)^{1/4} \int_{-\infty}^{\infty} \frac{dp}{\hbar}\, exp\left(\frac{-p^2a^2}{\hbar^2} - \frac{i(p-p_0)x}{\hbar}\right) .$$

(a) Define the position and momentum uncertainties Δx and Δp, and calculate them for this wave packet at time $t = 0$. Evaluate the quantity $\Delta x \Delta p$ at time $t = 0$ and determine whether it is less than, greater than or equal to $\hbar/2$. Explain the significance of your answer.

(b) The packet is now allowed to propagate in space for a time t. Determine $\Delta x(t)$ and $\Delta p(t)$. Have the ratios $\Delta x(t)/\Delta x(0)$, $\Delta p(t)/\Delta p(0)$, $(\Delta x(t)\Delta p(t))/(\Delta x(0)\Delta p(0))$ increased, decreased, or stayed the same? Explain this behavior.

2.6

(a) Consider a general ket $|\ell, m\rangle$ where ℓ designates the orbital angular momentum eigenvalue and m its z component. Consider a specific ket $|2,1\rangle$. Determine for which $|\ell, m\rangle$ values the matrix elements

$$\langle 2,1|r^2|\ell,m\rangle \quad , \quad \langle 2,1|r\mathbf{r}|\ell,m\rangle$$

are non-zero, and give their values.

(b) Consider the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Evaluate the commutator $[L_x, ay^2p_z^2 + br^2]$ where a and b are pure numbers.

(c) In a certain representation the angular momentum operator L_x is given by:

$$L_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} .$$

What angular momentum ℓ is associated with this form for L_x? What are the eigenvalues of this L_x?

(d) From the fact that $L_+|\ell\ell\rangle = 0$ construct an explicit form for $Y_\ell^\ell(\theta\phi)$.