Preliminary Exam : Quantum Physics

August 23, 2002, 9:00 a.m. - 1:00 p.m.

Please answer 3 QUESTIONS from each of the two sections.

Please use a separate book FOR EACH QUESTION.

Section I: Statistical Mechanics

1.) Consider a set of N noninteracting spin 1/2 particles, with magnetic moments m, placed in a constant magnetic field B.

(a) Use the partition function to find the internal energy U of the spin system in thermal equilibrium at temperature T.

(b) Show that the magnetization of the system is given by $M = n\,m\,\tanh[m\,B/(k_B T)]$, where $n = N/V$ is the spin concentration.

(c) Find the magnetic susceptibility in the high-temperature/low-field limit.

2.)

(a) Using the free electron model for a simple two-dimensional metal with n conduction electrons per unit area, show that the density of states per unit energy per unit area is given by

$$g(\epsilon) = \frac{m}{\pi\,\hbar^2}$$

(b) Calculate the Fermi energy $\epsilon_F = \epsilon_F(T = 0)$ for this two dimensional metal.

(c) Show that the chemical potential (or Fermi level) for this two dimensional metal is given by

$$\mu(T) = \epsilon_F(T) = k_B\,T\,\ln\left[\exp(\pi\,n\,\hbar^2/m\,k_B T) - 1\right]$$

3.) Consider a system of three-dimensional rigid rotators (with no translational motion), with moment of inertia I, in thermal equilibrium according to Boltzmann statistics. Taking into account the quantization of the energies,

(a) compute the **leading** term in the low temperature expansion of the specific heat C_V.

(b) Given that at high temperature the partition function behaves as

$$Z \approx \frac{2IkT}{\hbar^2} + \frac{1}{3} + O\left(\frac{1}{T}\right) \quad ,$$

compute the **leading** term in the high temperature expansion of C_V.

(c) Make a rough sketch of the T dependence of C_V, and comment on the physical significance of the leading behaviors of C_V at low and high T.

4.) Consider a perfect crystal of N identical atoms. Suppose that n atoms are displaced from the lattice sites to interstitial sites between the lattice sites. Assume that $1 \ll n \ll N$, and that N', the number of available interstitial sites, is of the same order of magnitude as N.

If W is the energy required to displace an atom from a lattice site to an interstitial site, and $kT \ll W$, show that in equilibrium,

$$n \approx \frac{\sqrt{NN'}}{e} e^{-\frac{W}{2kT}}$$

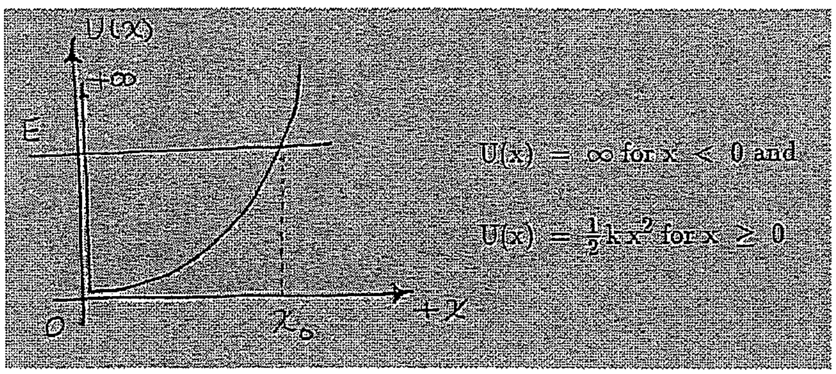
hint: recall Stirling's approximation, $\log(m!) \approx m \log(m) - m$, for large m

Section II: Quantum Mechanics

5.) Consider the WKB approximation for potentials with classical "turning points".

(a) Consider a particle with total energy E and mass m in a potential $U(x)$ in one dimension. Write down the general form of the WKB wavefunction for the particle : (i) in the classically allowed region, and (ii) in the classically forbidden region. Express your answers in terms of E, $U(x)$ and the appropriate constants.

(b) Consider the special case of the "half-harmonic oscillator" potential shown in the figure. Using the WKB approximation, obtain the bound state energies for this potential, and comment on their relation to the "full" harmonic oscillator energy levels.
(Note: $\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{\pi}{4}a^2$).



6.) (a) Consider the spin-orbit (SO) interaction correction to the Bohr spectrum of hydrogen. The perturbation is:

$$H'_{\rm SO} = \frac{e^2}{2m^2c^2r^3}\, \vec{S}\cdot\vec{L}$$

Show by careful dimensional arguments that this spin-orbit correction has order of magnitude equal to α^2 times the energy scale of the Bohr spectrum, where $\alpha = e^2/\hbar c$ is the fine-structure constant.
HINT: the Bohr radius is $a = \frac{\hbar^2}{me^2}$.

(b) The spin-orbit correction in (a), together with the first-order relativistic correction (which is of the same order of magnitude), constitute the "fine-structure correction" to the Bohr spectrum, giving the shifted energy levels:

$$E^{\rm FS}_{n,j} = E^{\rm Bohr}_n \left\{1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right\}$$

where j is the total angular momentum quantum number.

Into how many levels does the $n = 3$ Bohr level split at this order of fine-structure? What are the remaining degeneracies of these levels?

7.) Two spin $\frac{1}{2}$ Dirac fermions, one of mass m_1 and charge $+e$, and the other of mass m_2 and charge 0, are bound together in a non-relativistic bound state of spin j and magnetic moment μ.

(a) If orbital angular momentum l and spin s are individually good quantum numbers, what combinations of l and s are allowed for a state ψ_j such that $J^2\psi = j(j+1)\hbar^2\psi_j$.

(b) What are the values of μ for the four states with $j = 1$ and $m_j = 1$? (Note, as a self-check, that the four values of μ should converge to one value in the limit $m_2 \gg m_1$.)

Table 1: Values of Clebsch-Gordan coefficients for some selected angular momentum quantum numbers.

j	m_j	ℓ	m_ℓ	s	m_s	$\langle jm_j \vert \ell m_\ell\, sm_s \rangle$
0	0	1	1	1	-1	$\sqrt{1/3}$
		1	0	1	0	$-\sqrt{1/3}$
		1	-1	1	1	$\sqrt{1/3}$
1	0	1	1	1	-1	$\sqrt{1/2}$
		1	0	1	0	0
		1	-1	1	1	$-\sqrt{1/2}$
		2	1	1	-1	$\sqrt{3/10}$
		2	0	1	0	$-\sqrt{2/5}$
		2	-1	1	1	$\sqrt{3/10}$
2	0	1	1	1	-1	$\sqrt{1/6}$
		1	0	1	0	$\sqrt{2/3}$
		1	-1	1	1	$\sqrt{1/6}$
		2	1	1	-1	$\sqrt{1/2}$
		2	0	1	0	0
		2	-1	1	1	$-\sqrt{1/2}$
1	1	1	1	1	0	$\sqrt{1/2}$
		1	0	1	1	$-\sqrt{1/2}$
		2	2	1	-1	$\sqrt{3/5}$
		2	1	1	0	$-\sqrt{3/10}$
		2	0	1	1	$\sqrt{1/10}$
2	1	1	1	1	0	$\sqrt{1/2}$
		1	0	1	1	$\sqrt{1/2}$
		2	2	1	-1	$\sqrt{1/3}$
		2	1	1	0	$\sqrt{1/6}$
		2	0	1	1	$-\sqrt{1/2}$

8.) A spin 0 particle scatters from a short-range central potential $V(r)$ starting from an initial plane wave of momentum k. It is observed that there is a resonance in the $l = 1$ partial wave at momentum k_r. If all other partial waves can be neglected and the scattering is pure elastic, what is the observed differential cross-section $\frac{d\sigma}{d\Omega}(\theta, \phi)$ at the momentum k_r?

You may make use of the identity: $e^{ikz} = \sum_{l=0}^{\infty} i^l \, (2l+1) \, j_l(kr) \, P_l(\hat{r})$.