Preliminary Exam: Quantum Mechanics, Thursday January 16, 2020. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. Put the solution to each problem in a **separate** blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\boldsymbol{\nabla}\psi = \mathbf{e}_x\frac{\partial\psi}{\partial x} + \mathbf{e}_y\frac{\partial\psi}{\partial y} + \mathbf{e}_z\frac{\partial\psi}{\partial z} = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} = \mathbf{e}_\rho\frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z}.$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad , \quad H_0(x) = 1 \quad , \quad H_1(x) = 2x \quad , \quad H_2(x) = 4x^2 - 2$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right) \quad , \quad \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r) \quad .$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad , \quad P_0(x) = 1 \quad , \quad P_1(x) = x \quad , \quad P_2(x) = \frac{1}{2}(3x^2-1) \quad ,$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P^m_l(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y^m_l(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P^m_l(\cos\theta)e^{im\phi} \quad ,$$

$$Y^0_0 = \left(\frac{1}{4\pi}\right)^{1/2} \quad , \quad Y^0_1 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \quad , \quad Y^{\pm1}_1 = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y^0_2 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \ , \ Y^{\pm1}_2 = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \ , \ Y^{\pm2}_2 = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \quad , \quad n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \quad ,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} \quad , \quad n_\ell(r) \to \frac{\sin(r-\ell\pi/2-\pi/2)}{r} \quad .$$

$$j_0(r) = \frac{\sin r}{r} \quad , \quad n_0(r) = -\frac{\cos r}{r} \quad , \quad j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \quad , \quad n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \quad ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \quad , \quad n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \quad .$$

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta).$$

1. As a by-product of the proof of the minimum uncertainty principle, it may be shown that if two observables A and B satisfy $(\Delta A)^2\,(\Delta B)^2 = \frac{1}{2}|\langle[A,B]\rangle|^2$ in a state $\psi(x)$, then the states $(A-\langle A\rangle)\psi$ and $(B-\langle B\rangle)\psi$ are linearly dependent, i.e., one is a constant multiple of the other. The latter is therefore a necessary condition for the uncertainty principle to be satisfied as an equality. Henceforth consider position and momentum observables x and p in states $\psi(x)$ that are normalized to one, and denote the corresponding expectation values by x_0 and p_0.

   (**a**) Show that all states that satisfy $\Delta x\,\Delta p = \frac{1}{2}\,\hbar$ as an equality must be of the form

   $$\psi(x) = C\exp\left[\alpha(x-x_0)^2 + i\frac{p_0 x}{\hbar}\right] \tag{1}$$

   for some complex constants C and α.

   (**b**) However, the necessary condition (1) is not sufficient: Show that if $\Delta x\,\Delta p = \frac{1}{2}\,\hbar$ is to hold true, the real part of α must be negative and the imaginary part must equal zero.

   The result effectively says that Gaussian wave packets of the form

   $$\psi(x) = \frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left[-\frac{(x-x_0)^2}{4\sigma^2} + i\frac{xp_0}{\hbar}\right]$$

   with some real x_0, p_0 and $\sigma > 0$ are the only minimum-uncertainty wave packets.

   Hint: Using momentum-space wave functions may help. Recall the integral

   $$\int_{-\infty}^{\infty} dx\,\exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}}\,\exp\left(\frac{\beta^2}{4\alpha}\right)$$

   valid for all complex α and β with $\mathrm{Re}(\alpha) > 0$; otherwise the integral diverges.

2. Consider a two-level system with Hamiltonian

   $$\frac{H}{\hbar} = -\tfrac{1}{2}\Delta\,(\,|2\rangle\langle 2| - |1\rangle\langle 1|\,) - \tfrac{1}{2}\kappa(\,|1\rangle\langle 2| + |2\rangle\langle 1|\,)\,.$$

   This could describe two states in an atom coupled by laser light, so that the "detuning" parameter Δ is the difference between the laser frequency and the atomic resonance frequency, and the "Rabi frequency" κ is determined by the electric field amplitude of the laser E and the appropriate dipole moment matrix element d as $\kappa = dE$.

   (**a**) Show that the eigenfrequencies are $\Omega_\pm = \pm\frac{1}{2}\sqrt{\Delta^2+\kappa^2}$

   (**b**) Now define (this obviously is possible) an angle θ such that

   $$\cos\theta = \frac{\Delta}{\sqrt{\Delta^2+\kappa^2}}, \qquad \sin\theta = -\frac{\kappa}{\sqrt{\Delta^2+\kappa^2}}\,.$$

   Show that the vectors

   $$|+\rangle = \cos\frac{\theta}{2}\,|1\rangle + \sin\frac{\theta}{2}\,|2\rangle, \quad |-\rangle = -\sin\frac{\theta}{2}\,|1\rangle + \cos\frac{\theta}{2}\,|2\rangle$$

   are unit-normalized eigenvectors of the Hamiltonian corresponding to the eigenvalues $\Omega_\pm$.

   (**c**) Suppose the detuning Δ is time dependent: that it starts from some very large (in absolute value) negative value, and is swept so that it ends up with a very large positive value. The system starts in the state $|1\rangle$. What will be the state after the sweep if the sweep is (i) very fast, or (ii) very slow? The latter case is called "rapid adiabatic passage."

3. The Hamiltonian of a harmonic oscillator is written in terms of creation and annihilation operators as

$$\hat{H}_0 = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}).$$

A coherent state $|\alpha\rangle$ is defined as the eigenstate of the annihilation operator with (complex) eigenvalue α:

$$\hat{a}|\alpha\rangle \;=\; \alpha|\alpha\rangle.$$

(a) Show that $|\alpha\rangle$ can be written in terms of the harmonic oscillator eigenstates (i.e. eigenstates that obey $\hat{n}|n\rangle = \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle$, where n is a non-negative integer) in the following form

$$|\alpha\rangle \;=\; \mathcal{C}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

What value for the factor $\mathcal{C}$ is needed to fix the normalization of $|\alpha\rangle$ to one?

(b) Show explicitly that the following relations hold:

$$\exp\left(-\frac{i}{\hbar}\hat{H}_0 t\right)\hat{a}\exp\left(\frac{i}{\hbar}\hat{H}_0 t\right) \;=\; e^{i\omega t}\hat{a}, \qquad \exp\left(-\frac{i}{\hbar}\hat{H}_0 t\right)\hat{a}^\dagger\exp\left(\frac{i}{\hbar}\hat{H}_0 t\right) \;=\; e^{-i\omega t}\hat{a}^\dagger.$$

(c) Show that for a general unitary operator $\hat{U}$ the following relation is true for any operator $\hat{A}$:

$$\hat{U}\exp(\hat{A})\hat{U}^{-1} \;=\; \exp\left(\hat{U}\hat{A}\hat{U}^{-1}\right).$$

The coherent state can also be written as the displacement from the harmonic oscillator vacuum state as

$$|\alpha\rangle \;=\; \hat{\mathcal{D}}(\alpha)|0\rangle,$$

where $\hat{\mathcal{D}}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$. (No need to prove this.) Consider the evolution operator $\hat{U}_0 = e^{-\frac{i}{\hbar}H_0 t}$. Calculate $\hat{U}_0\hat{\mathcal{D}}(\alpha)\hat{U}_0^{-1}$ in order to find the time-dependent state vector $|\psi(t)\rangle$ that evolves from a coherent state at $t = 0$. Show that $|\psi(t)\rangle$ is also a coherent state.

4. Assume you have two observables $(\hat{\Omega}, \hat{\Lambda})$ that are compatible (i.e., they have a common set of eigenstates). Further assume that each eigenvalue (out of several) of either observable is degenerate, meaning that for each eigenvalue ω of $\hat{\Omega}$ there are several eigenstates, and similarly for $\hat{\Lambda}$. However, also assume that each pair of eigenvalues (ω, λ) uniquely defines one and only one state of the system (except for multiplication by a complex number), so that the set of eigenvectors $|\omega, \lambda\rangle$ is a complete and orthonormal basis of the Hilbert space. Initially, the system is in some state $|\psi\rangle$ that is completely unknown to me. I first measure observable $\hat{\Omega}$, with result ω_1, immediately followed by a measurement of $\hat{\Lambda}$ with result λ_1, and then by a second measurement of $\hat{\Omega}$.

(a) Describe the extent of my knowledge about which state the system is in after each of the three measurements. Be as precise as possible: What do I know and what do I not know about the state at each point?

(b) At which point along this chain do I know everything there is to know about the (present) state of the system?

(c) For which of the three measurements can I predict the exact outcome?

(d) For which of the same three measurements can I predict the probability of a given possible outcome?

Now assume that I *do* completely know the state $|\psi\rangle$ of the system initially (before the first measurement), but it is not an eigenstate of either $\hat{\Omega}$ or $\hat{\Lambda}$. Answer the same four questions again for this case.

5. A one-dimensional harmonic oscillator has an unperturbed Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}mw_0^2 q^2,$$

where ω_0 is its natural frequency. At time $t = 0$ it is in its nth energy eigenstate $|n\rangle$. It is then perturbed by a potential $V(t) = -Aq^3\sin(\omega t)$ where A is a small constant.

(a) To what states does the state $|n\rangle$ make transitions to lowest order in the perturbation, and what are the associated transition probability amplitudes (i.e. matrix elements)?

(b) List all transitions from $|n\rangle$ that are possible in second order.

(c) The harmonic oscillator is in its ground state $|n = 0\rangle$ at time $t = 0$ when the above $V(t)$ perturbation is applied. If the perturbation is switched off at time $t = t_0$, what is the first order probability of still finding the oscillator in its ground state at times $t \gg t_0$.

(d) Discuss briefly how the probability in part (**c**) would change if the frequency ω of the perturbation $V(t)$ just happened to be equal to the natural frequency ω_0 of the oscillator.
Hint: You might find it useful to set

$$q = \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger), \quad p = i\sqrt{\frac{\hbar m\omega_0}{2}}(a^\dagger - a)$$