Preliminary Exam: Quantum Mechanics, Friday January 18, 2019, 9:00am-1:00pm

Answer a total of any **FOUR** out of the five questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \ H_0(x) = 1, \ H_1(x) = 2x, \ H_2(x) = 4x^2 - 2.$$

$$\text{Laguerre} = L_n(r) = e^r \frac{d^n}{dr^n}\left(r^n e^{-r}\right), \quad \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q \frac{d^q}{dr^q} L_{n+q}(r).$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l, \ P_0(x) = 1, \ P_1(x) = x, \ P_2(x) = \frac{1}{2}(3x^2-1),$$

$$\int_{-1}^{+1} dw P_\ell(w) P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}.$$

$$\text{associated Legendre polynomial} = P^m_l(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}} P_l(x)$$

$$\text{spherical harmonic} = Y^m_l(\theta, \phi) = (-1)^m \left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P^m_l(\cos\theta) e^{im\phi} ,$$

$$Y^0_0 = \left(\frac{1}{4\pi}\right)^{1/2} , \ Y^0_1 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta , \ Y^{\pm 1}_1 = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$$

$$Y^0_2 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1) , \ Y^{\pm 1}_2 = \mp \left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta e^{\pm i\phi} , \ Y^{\pm 2}_2 = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \ j_l(r) = (-1)^\ell r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\sin r}{r}\right) , \ n_l(r) = (-1)^{(\ell+1)} r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\cos r}{r}\right) ,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} , \ n_\ell(r) \to \frac{\sin(r - \ell\pi/2 - \pi/2)}{r} .$$

$$j_0(r) = \frac{\sin r}{r} , \ n_0(r) = -\frac{\cos r}{r} , \ j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} , \ n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} , \ n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} .$$

1. An electric field $E(t)$ (such that $E(t) \to 0$ fast enough as $t \to -\infty$) is incident on a charged (q) harmonic oscillator (ω) in the x direction, which gives rise to an added "potential energy" $V(x,t) = -qxE(t)$. This whole problem is one-dimensional.

   (a) Using first-order time dependent perturbation theory, write down the amplitude $c_n(t)$ for finding the system in excited state n at time t if the system starts in $n = 0$ at $t = -\infty$.

   (b) Find the expectation value of the momentum of the oscillator as a function of time.

   (c) The exact classical solution to the same problem with the initial conditions $x = p = 0$ at $t = -\infty$ is

   $$p(t) = \mathrm{Re}\left[\int_{-\infty}^{t} dt'\, e^{-i\omega(t-t')} qE(t')\right].$$

   Compare this with the result of part (b).

2. Take two hermitian operators $\hat{x}$ and $\hat{p}$ such that $[\hat{x}, \hat{p}] = i\hbar$. The spectrum of $\hat{x}$ is continuous and we have $\hat{x}|x\rangle = x|x\rangle$, $x \in \mathbb{R}$. Also, the orthonormal basis states are normalized according to $\langle x'|x\rangle = \delta(x' - x)$.

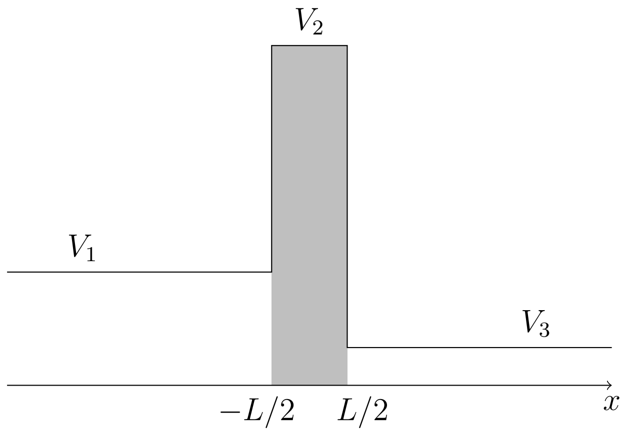
   (a) Show that $[\hat{x}, \hat{p}^n] = i\hbar\, n\hat{p}^{n-1}$ holds true for positive integers n, and that $\left[\hat{x}, e^{-i\frac{\lambda\hat{p}}{\hbar}}\right] = \lambda e^{-i\frac{\lambda\hat{p}}{\hbar}}$. Henceforth assume that $\lambda \in \mathbb{R}$.

   (b) Show that $e^{-i\frac{\lambda\hat{p}}{\hbar}}|x\rangle$ is an eigenvector of the operator $\hat{x}$ with the eigenvalue $x + \lambda$. In fact, it is possible to choose the phase factors of the basis states $|x\rangle$ so that $e^{-i\frac{\lambda\hat{p}}{\hbar}}|x\rangle = |x + \lambda\rangle$; assume this has been done.

   (c) Show that $\langle x|\hat{p}|x'\rangle = \frac{\hbar}{i}\delta'(x - x')$, proportional to the derivative of the delta function. Hint: express $\hat{p}$ as a derivative of $e^{-i\frac{\lambda\hat{p}}{\hbar}}$.

   (d) What does the result of part (c) say about momentum operator in position representation?

3. A potential barrier V_2 of width L is centered around $x = 0$ between regions with constant potential V_1 for $x \leq -L/2$ and V_3 for $x \geq L/2$ with $V_3 < V_1$. The subscripts are used to consistently label the three regions with different potentials.



A particle of mass m is sent in from $x = -\infty$, traveling towards the right with energy E. The wave-function of the incoming particle is $\psi_{in} = Ae^{ik_1x}$

(a) What are the wave-functions in the three different regions for $E < V_2$?

(b) What boundary conditions must be satisfied?

(c) Calculate the transmission coefficient of the barrier.

(d) Where is the particle traveling fastest?

(e) What are the effects of an additional barrier of width w and height V_2 centered at $x = 3L$ while the potential at $x > 3L + w/2$ is the same as V_3? Is the presence of this additional barrier sufficient to permit a bound state and why?

4. Let a, $a^\dagger$ be the annihilation and creation operators,

$$[a; a^\dagger] = 1.$$

Consider a harmonic oscillator which in suitable units takes the form

$$H = \hbar\omega(a^\dagger a + 1/2) + C(a + a^\dagger)$$

where C is a real constant.

(a) Compute the ground state energy of H to second order in perturbation theory in C.

(b) Find the exact spectrum of H. (*Hint:* Define new creation and annihilation operators b and $b^\dagger$, related to the original a and $a^\dagger$ by a constant shift, and notice that this maps the Hamiltonian to one for which the spectrum is obvious.)

(c) Assuming that a is related to the position operator x and momentum operator p by $a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{ip}{m\omega})$, find the ground state wave function of H. Give a physical interpretation of the overall effect that the term proportional to C has on the eigenstates and the energy spectrum for this system.

5. An elementary spin-1/2 particle with electric charge q and mass m interacts with classical electromagnetic fields according to the Dirac Hamiltonian $H = c\vec{\alpha} \cdot \vec{\pi} + \beta mc^2 + q\Phi$ where the operator $\vec{\pi} = \vec{p} - q\vec{A}$, with $\vec{A}$ representing the electromagnetic vector potential and Φ the scalar potential. Explicit forms for the four 4×4 matrices $\vec{\alpha}$ and β are given below. In the non-relativistic limit the 4-component Dirac spinor $\psi(t, \vec{x})$ can be written as two 2-component Pauli spinors, the "large" χ_ℓ and "small" χ_s components:

$$\psi(t, \vec{x}) = \begin{pmatrix} \chi_\ell(t, \vec{x}) \\ \chi_s(t, \vec{x}) \end{pmatrix} e^{-imc^2 t/\hbar}.$$

(a) Show that in the non-relativistic limit $\chi_s = \frac{1}{2mc}(\vec{\sigma} \cdot \vec{\pi})\chi_\ell$. Use this result to show that the norm χ_s is much smaller than the norm of χ_ℓ in this limit.

(b) Show that in the non-relativistic limit χ_ℓ obeys the so-called Pauli equation:

$$i\hbar \frac{\partial}{\partial t} \chi_\ell(t, \vec{x}) = \left(\frac{(\vec{\sigma} \cdot \vec{\pi})^2}{2m} + q\Phi \right) \chi_\ell(t, \vec{x}).$$

(c) Starting from the Pauli equation given in part (b), show that the interaction of the particle's spin with the external magnetic field is described by the interaction $H_{\text{int}} = -\vec{\mu} \cdot \vec{B}$. Derive the expression for the magnetic moment operator $\vec{\mu}$.

Remarks: The Dirac matrices α_i for $i = 1, 2, 3$ and β are given in terms of the 2×2 Pauli matrices σ_i and the 2×2 unit matrix $\mathbb{1}$ as follows:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Hint: In the non-relativistic limit it can be assumed that $i\hbar \frac{\partial}{\partial t} \chi_i(t, \vec{x})$ is negligible with respect to $mc^2 \chi_i(t, \vec{x})$ for $i = \ell$, s and $q\Phi$ can be neglected with respect to mc^2.