Preliminary Exam: Quantum Mechanics, Friday January 12, 2018. 9:00-1:00

Answer a total of any **FOUR** out of the five questions. Put the solution to each problem in a separate blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than four problems, only the first four problems as listed on the exam will be graded.

Some possibly useful information

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\boldsymbol{\nabla}\psi = \mathbf{e}_x\frac{\partial\psi}{\partial x} + \mathbf{e}_y\frac{\partial\psi}{\partial y} + \mathbf{e}_z\frac{\partial\psi}{\partial z} = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} = \mathbf{e}_\rho\frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z}.$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad , \quad H_0(x)=1 \quad , \quad H_1(x)=2x \quad , \quad H_2(x)=4x^2-2$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right) \quad , \quad \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r) \quad .$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad , \quad P_0(x)=1 \quad , \quad P_1(x)=x \quad , \quad P_2(x)=\frac{1}{2}(3x^2-1) \quad ,$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\ \phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi} \quad ,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \quad , \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \quad , \quad Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \ , \ Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \ , \ Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \quad , \quad n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \quad ,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r-\ell\pi/2-\pi/2)}{r} \quad , \quad n_\ell(r) \to \frac{\sin(r-\ell\pi/2-\pi/2)}{r} \quad .$$

$$j_0(r) = \frac{\sin r}{r} \quad , \quad n_0(r) = -\frac{\cos r}{r} \quad , \quad j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \quad , \quad n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \quad ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \quad , \quad n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \quad .$$

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta).$$

1. (**a**) Consider the addition of two general angular momentum operators according to $\mathbf{L}_1 + \mathbf{L}_2 = \mathbf{L}$. Eigenstates $|\ell_1, m_1\rangle$ are associated with the operators $\mathbf{L}_1^2$ and L_{1z}, eigenstates $|\ell_2, m_2\rangle$ are associated with the operators $\mathbf{L}_2^2$ and L_{2z}, and eigenstates $|L, M\rangle$ are associated with the operators $\mathbf{L}^2$ and L_z. In terms of the quantum numbers (ℓ_1, m_1) and (ℓ_2, m_2) state (i.e. no need to derive) the values which are allowed for the quantum numbers (L, M).

   (**b**) Consider (**a**) in the specific case in which $L_1 = 1$ and $L_2 = 2$. In terms of the basis vectors $|\ell_1, m_1\rangle$ and $|\ell_2, m_2\rangle$ appropriate to this case, construct all the relevant eigenstates $|L, M\rangle$ which possess the two highest allowed positive M values.

   (**c**) Consider a general state $|\ell, m\rangle$ where ℓ designates the orbital angular momentum eigenvalue, and m its z component, Determine for which $|\ell, m\rangle$ values the matrix elements

$$\langle 1, 0|r^2|\ell, m\rangle, \qquad \langle 1, 0|r\mathbf{r}|\ell, m\rangle$$

   are non-zero, and determine their values. There is no need to include any radial wave function or do any radial integrations.

2. The differential cross-section in a certain scattering process is known to be given by

$$\sigma(\theta) = \alpha + \beta\cos(\theta) + \gamma\cos^2(\theta).$$

   (**a**) What is the scattering amplitude?

   (**b**) Express α, β, γ in terms of the phase shifts.

   (**c**) Are there any constraints on the magnitudes of α, β and γ if the scattering amplitude is not allowed to grow any faster than $\ln E$ as the energy E becomes very large?

   (**d**) Deduce the total scattering cross-section and show that it is consistent with the optical theorem.

3. What do you understand by parity? The parity π_A of an operator $\hat{A}$ is defined using $P\hat{A}P^{-1} = \pi_A\hat{A}$ and the parity π_ψ of a state vector $|\psi\rangle$ is defined using $P|\psi\rangle = \pi_\psi|\psi\rangle$ (if and when they exist).

   (**a**) Show that the selection rule

$$\pi_\alpha \pi_A \pi_\beta = 1$$

   applies to the matrix element $\langle\alpha|A|\beta\rangle$ (i.e. this matrix element has to be zero when the above product of parities is not equal to 1).

   (**b**) Deduce that a nucleon dipole moment in a state $|\psi\rangle$ defined as $\langle\psi|\mathbf{d}|\psi\rangle$ has to be zero where $\mathbf{d}$ $(= q\mathbf{r})$ is the dipole moment operator.

   (**c**) Show explicitly how you would determine the parity of a spherical harmonic $Y_{\ell m}(\theta, \phi)$.

4. The time-dependent Schrodinger equation for a non-relativistic charged particle moving in a magnetic field is given by the formula:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \hat{H}\psi(\mathbf{r},t), \qquad \hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A}(\mathbf{r},t))^2 + q\phi(\mathbf{r},t)$$

where $\psi(\mathbf{r},t)$ is the particle wave function, $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$ is the momentum operator, m and q are the particle mass and charge, and $\phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ are the scalar and vector potentials respectively.

(**a**) Derive from the Schrodinger equation the following analytic expression for the probability current density $\mathbf{j}(\mathbf{r},t)$:

$$\mathbf{j}(\mathbf{r},t) = \frac{i\hbar}{2m}\left[\psi(\mathbf{r},t)\boldsymbol{\nabla}\psi^*(\mathbf{r},t) - \psi^*(\mathbf{r},t)\boldsymbol{\nabla}\psi(\mathbf{r},t)\right] - \frac{q}{m}\mathbf{A}(\mathbf{r},t)|\psi(\mathbf{r},t)|^2,$$

taking into account that the probability density $\rho(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ and the current density $\mathbf{j}(\mathbf{r},t)$ satisfy the continuity equation

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j}(\mathbf{r},t) = 0$$

(**b**) Show that the current density $\mathbf{j}(\mathbf{r},t)$ is an invariant under the gauge transformation

$$\mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \boldsymbol{\nabla}f(\mathbf{r},t), \qquad \phi'(\mathbf{r},t) = \phi(\mathbf{r},t) - \frac{\partial}{\partial t}f(\mathbf{r},t)$$

for arbirtrary $f(\mathbf{r},t)$.

Hint: The gauge transformation of the wave function has to be taken into account. In this problem the velocity of light c is taken to be $c = 1$.

5. A non-relativistic particle with mass m is bound by a deep, spherically symmetric potential of small radius r_0, with the center of the potential being located at $\mathbf{R}_j$. The particle Hamiltonian $\hat{H} = -(\hbar^2/2m)\boldsymbol{\nabla}^2$ describes free particle motion in the entire region, except only for the small $r < r_0$ region inside the potential well. The well radius is allowed to go to zero ($r_0 \to 0$), but the well is so deep that it can support a single discrete energy level with angular momentum $\ell = 0$. In the $r_0 \to 0$ limit the effect of potential well can be replaced by a boundary condition for the particle wave function $\psi(\mathbf{r})$ of the form:

$$\psi(\mathbf{r} \to \mathbf{R}_j) = C\left[\frac{1}{r_j} - \beta_j + O(r_j)\right]$$

where $r_j = |\mathbf{r} - \mathbf{R}_j|$ is the distance between the particle and the center of the well at $\mathbf{R}_j$, C is a constant, and β_j is a positive constant.

(**a**) Calculate the bound state wave function $\psi(\mathbf{r})$ and energy ϵ_1 of the particle moving in the potential with parameters β_1 and $\mathbf{R}_1 = 0$.

(**b**) Derive the secular equation that determines the allowed energies ϵ of discrete states if the particle is bound by two short-range potentials with parameters β_1 at $\mathbf{R}_1 = 0$ and β_2 at $\mathbf{R}_2$.

Hint: The eigenfunction $\psi_2(\mathbf{r})$ of the particle in the presence of the two short-range potentials can be represented by a linear combination of two functions that are centered at $\mathbf{R}_1 = 0$, $\mathbf{R}_2 = \mathbf{R}$:

$$\psi_2(\mathbf{r}) = A\psi(\mathbf{r}) + B\psi(\mathbf{r} - \mathbf{R})$$

where A and and B are constants.