Preliminary Examination: Quantum Mechanics, 1/13/2012

Answer a total of **FOUR** questions out of **FIVE**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are, and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

Possibly Useful Information

$$\int_{-\infty}^{+\infty} dx\ \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}} \exp(\frac{\beta^2}{4\alpha}) \text{ with } \mathrm{Re}(\alpha) > 0, \quad \int_0^{\infty} dx\ x\ \exp(-\alpha x^2) = \frac{1}{2\alpha}$$

First few spherical harmonics

$$Y_{00} = (\frac{1}{4\pi})^{1/2}, \quad Y_{10} = (\frac{3}{4\pi})^{1/2} \cos\theta, \quad Y_{1\pm1} = \mp(\frac{3}{8\pi})^{1/2} \sin\theta\ \exp(\pm i\phi)$$

$$Y_{20} = (\frac{5}{16\pi})^{1/2}\ (3\cos^2\theta - 1), \quad Y_{2\pm1} = \mp(\frac{15}{8\pi})^{1/2}\ \sin\theta\cos\theta\ \exp(\pm i\phi),$$

$$Y_{2\pm2} = (\frac{15}{32\pi})^{1/2}\ \sin^2\theta\ \exp(\pm 2i\phi)$$

Legendre polynomials

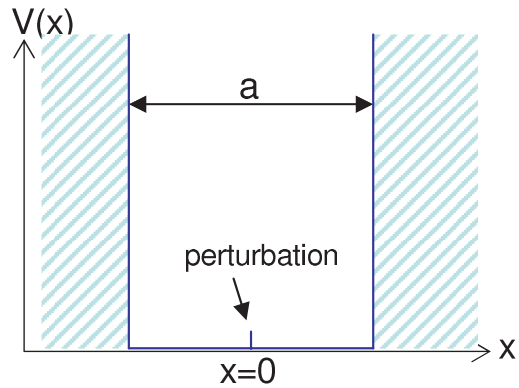
$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

QM1. A particle of mass m is initially in the ground state of an infinite square well of width a, as shown in the figure. At $t = 0$ a time-varying potential is turned on that is given by

$$V(t) = V_0\,\delta(x)\sin(\omega_D t)$$

which drives the system at the fixed frequency ω_D. Consider the case where V_0 is small enough for $V(t)$ to be considered a time-dependent perturbation.



(a) Write down the energies E_0, E_1, E_2, and wavefunctions ϕ_1, ϕ_2, ϕ_3 of the ground state and the first and second excited states of the unperturbed system.

(b) Starting from the time dependent Schrödinger equation, show that the amplitude for the system being found in excited state n at time $t > 0$ is given by the following expression to leading order in the perturbation,

$$c_n(t) = -\frac{i}{\hbar}\int_0^t dt'\, e^{i(\omega_n-\omega_0)t'}V_{n0}(t')\,c_0(0)\,,$$

where $V_{nn'}(t) = \langle\phi_n|V|(t)|\phi_{n'}\rangle$.

(c) What is the probability that a measurement of the energy of the system at time t would yield the result E_2 to leading order in V?

QM2. Consider measurements of a spin 1/2 in the direction $\hat{\mathbf{n}} = \cos\alpha\,\hat{\mathbf{e}}_x + \sin\alpha\,\hat{\mathbf{e}}_z$, so that we are measuring the operator $S(\alpha)$ with the corresponding unit-normalized eigenvectors $\chi_\pm(\alpha)$,

$$S(\alpha) = \frac{\hbar}{2}\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix};\ \chi_+(\alpha) = \begin{bmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{bmatrix},\ \chi_-(\alpha) = \begin{bmatrix} -\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} \end{bmatrix}.$$

(a) Suppose you prepare the system initially in the state $\chi_+(0)$, then measure successively in the directions α, 2α, $\ldots$, $n\alpha$. Show that the probability that the result is $+\hbar/2$ every time is $[\cos\frac{\alpha}{2}]^{2n}$.

(b) Now make the angle α smaller and the number of measurements larger in such a way that $n\alpha = \pi$ remains constant. What do you achieve in the limit $n \to \infty$?

QM3. Let us study an isotropic (m, ω) harmonic oscillator in the xy plane. The oscillator is also rotated at an angular velocity $\Omega\hat{\mathbf{e}}_z$ about an axis that goes through the origin, with the usual result that in the rotating frame the Hamiltonian is amended with the term $-\Omega L_z$. Given the usual ladder operator for the x and y direction $a = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{i}{m\omega}p_x)$ and $b = \sqrt{\frac{m\omega}{2\hbar}}(y + \frac{i}{m\omega}p_y)$, the Hamiltonian in the rotating frame is

$$\frac{H}{\hbar} = \omega(a^\dagger a + b^\dagger b) + i\Omega(a^\dagger b - b^\dagger a)\,.$$

In this problem the zero point energies can be, and are, ignored. Let $|n_x, n_y\rangle$ denote the eigenstate of the "unperturbed" $(\Omega = 0)$ oscillator.

(a) The lowest excited state of the unperturbed oscillator is a degenerate manifold with the energy $\hbar\omega$ spanned by the states $\{|0,1\rangle, |1,0\rangle\}$. Find the energies of the two lowest-energy excited states in the rotating frame by treating the rotation as a small perturbation.

Now introduce the operators $\alpha = \frac{1}{\sqrt{2}}(a + i\,b)\,, \beta = \frac{1}{\sqrt{2}}(a - i\,b)\,.$

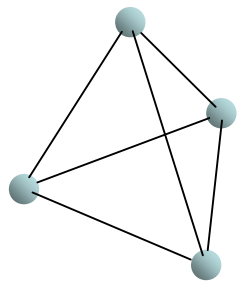
(b) Show that these operators are legitimate ladder operators for two independent harmonic oscillators.

(c) Express the total Hamiltonian in terms of the operators α and β.

(d) What does a comparison of the results of parts (a) and (c) tell you about higher orders of perturbation theory?

QM4. Four massive spin-1/2 particles are fixed on the vertices of a regular tetrahedron, as depicted in the figure. The Hamiltonian for the system consists of a sum of spin-spin interactions over each of the six pairs as follows,

$$\mathcal{H} = \alpha \sum_{i \neq j} \vec{S}_i \cdot \vec{S}_j.$$



(a) Show that all three components of the total spin $\vec{J} = \sum_i \vec{S}_i$ of the system commutes with $\mathcal{H}$.

(b) List all of the allowed energy levels for the system, and the degeneracy factors of each level. Please note, you do not have to construct any eigenstates of energy explicitly.

QM5. The rotational spectrum of a diatomic molecule consists of a sequence of lines with some energy spacing. In the rigid rotor approximation, its Hamiltonian can be expressed as $\mathcal{H} = \mathbf{L}^2/2I$ where $\mathbf{L}$ is the angular momentum operator and I is the moment of inertia about its center of mass or, equivalently, that of a hypothetical single particle of reduced mass μ rotating about a fixed axis at a distance equal to the bond length of the molecule.

(a) What are the energy eigenvalues and eigenfunctions of $\mathcal{H}$?

(b) Assume that the absorption transitions occur between adjacent levels. It turns out that the absorption spectrum consists of a series of equally spaced lines with constant spacing B. Express the bond length r_0 of the molecule in terms of this spacing B in the rigid rotor approximation.

(c) At time $t = 0$, the molecule is prepared in the superposition state

$$\Psi = \frac{Y_{00} + Y_{10}}{\sqrt{2}}.$$

This state evolves to $\Psi(t)$ at time t. Show that $|\Psi(t)|^2$ is a periodic function of time and determine its period T.