Preliminary Exam: Quantum Physics 1/13/2006, 9:00-3.00

Answer a total of **SIX** questions of which at least **TWO** are from section **A**, and at least **THREE** are from section **B**. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set.

Some possibly useful information:

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\int_0^\infty dx\, e^{-a^2x^2} = \frac{\pi^{1/2}}{2a},\quad \int_0^\infty dx\, xe^{-a^2x^2} = \frac{1}{2a^2},$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad ,\quad H_0(x)=1 \quad , \quad H_1(x)=2x \quad ,\quad H_2(x)=4x^2-2$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right) \quad , \quad \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r) \quad .$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad , \quad P_0(x)=1 \quad , \quad P_1(x)=x \quad , \quad P_2(x)=\frac{1}{2}(3x^2-1) \quad ,$$

$$\int_{-1}^{+1} dw P_\ell(w)P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\,\phi) = (-1)^m\left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}P_l^m(\cos\theta)e^{im\phi} \quad ,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \quad , \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \quad , \quad Y_1^{\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1) \ , \ Y_2^{\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \ , \ Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels}: \quad j_l(r) = (-1)^\ell r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\sin r}{r}\right) \quad , \quad n_l(r) = (-1)^{(\ell+1)}r^\ell\left(\frac{1}{r}\frac{d}{dr}\right)^\ell\left(\frac{\cos r}{r}\right) \quad ,$$

$$\text{with asymptotic behavior} \quad j_\ell(r)\to\frac{\cos(r-\ell\pi/2-\pi/2)}{r} \quad , \quad n_\ell(r)\to\frac{\sin(r-\ell\pi/2-\pi/2)}{r} \quad .$$

$$j_0(r)=\frac{\sin r}{r} \quad , \quad n_0(r) = -\frac{\cos r}{r} \quad , \quad j_1(r)=\frac{\sin r}{r^2}-\frac{\cos r}{r} \quad , \quad n_1(r) = -\frac{\cos r}{r^2}-\frac{\sin r}{r} \quad ,$$

$$j_2(r) = \frac{3\sin r}{r^3}-\frac{\sin r}{r}-\frac{3\cos r}{r^2} \quad , \quad n_2(r) = -\frac{3\cos r}{r^3}+\frac{\cos r}{r}-\frac{3\sin r}{r^2} \quad ,$$

A convenient representation of the Dirac matrices is given as:

$$\alpha_x = \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix} \ , \ \alpha_y = \begin{pmatrix} 0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix} \ , \ \alpha_z = \begin{pmatrix} 0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix} \ , \ \beta = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

Section 1: Statistical Mechanics

A.1 A large number N of particles of mass m are dispersed in air taken to be at constant temperature T in the presence of the earth's gravitational field.

(a) Calculate the distribution of particles as a function of height h from the surface of the earth. You can assume that h is much smaller than the radius of the earth.

(b) Calculate the specific heat of these particles.

A.2 Consider an N-particle free electron gas in one dimension confined within a line of length L.

(a) For this system derive the density of states $D(E)$ as function of energy E.

(b) For this system calculate the Fermi energy E_F .

(c) For this system calculate the average energy per particle at zero temperature.

A.3 Consider a degenerate gas of N non-interacting electrons in a volume V at $T = 0°\text{K}$.

(a) Find an equation relating the pressure, energy and volume of this gas in the extreme relativistic limit in which the electron mass is ignored.

(b) For a gas of electrons with non-zero mass m, find the condition on N and V for which the result obtained in part (a) is approximately valid.

A.4 Consider a gas of spinless Bose particles of mass m occupying a volume V at a temperature T.

(a) Write down an integral expression which implicitly determines the chemical potential $\mu(T)$. Determine in which direction $\mu(T)$ moves as T is decreased.

(b) Find an expression for the Bose-Einstein transition temperature T_C, below which one must have macroscopic occupation of some single-particle state. Leave your answer in terms of the dimensionless integral which this expression contains.

(c) Find an expression for the total energy $U(T, V)$ for $T < T_C$.

Section 2: Quantum Mechanics

B.1 A one-dimensional harmonic oscillator in its ground state is subjected to the action of an external perturbative potential of the form to

$$V(x,t) = \frac{Ax^3}{\tau\sqrt{\pi}}e^{-t^2/\tau^2}$$

where A and τ are constants. The potential is switched on at time $t = -\infty$. To lowest order in $V(x,t)$ calculate the probability of finding the harmonic oscillator in its first excited state at time $t = \infty$. Can transitions to any other excited states of the oscillator be found, and if so to which ones?

B.2 Consider an ion which consists of a nucleus with charge Z and two electrons bound to it by Coulomb forces. For simplicity take the electrons to be spinless.

(a) If we were to ignore the direct interaction between the two electrons, what would then be the ground state energy and ground state wave function of the ion. Hint: The ground state wave function and ground state energy of an ionized atom with Z protons and one electron are of the form $\psi(\mathbf{r},t) = (Z/a_0)^{3/2}e^{-Zr/a_0}/\sqrt{\pi}$ and $E = -Z^2e^2/2a_0$ where $a_0 = \hbar^2/me^2$.

(b) Calculate the amount by which this ground state energy would shift to lowest order in a perturbation due to the direct interaction between the two electrons. Hint: Use the relation

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\,\theta)$$

where $r_<$ is the smaller and $r_>$ is the larger of $|\mathbf{r}_1|$ and $|\mathbf{r}_2|$.

(c) How much energy is required to remove one of the two electrons from the ion so as to ionize it into a system consisting of a nucleus with charge Z to which only one electron is bound.

B.3 At a given time $t = 0$ you are given a free rigid rotator with Hamiltonian $H = \mathbf{L}^2/2I$ (I is the moment of inertia) in the state

$$\psi(\theta,\phi,0) = A\sin^2\theta\cos(2\phi) + B\sin\,\theta\sin\,\phi \qquad (A \text{ and } B \text{ are constants}).$$

(a) For this state what values of the angular momentum operators $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$ and L_z will measurement find, and with what probabilities will these values occur.

(b) Into what state $\psi(\theta,\phi,t)$ will the above state $\psi(\theta,\phi,0)$ evolve at some general time t.

B.4 Calculate the average values of the operators $x^2p_x^3$ and $x^2p_x^2$ for a one-dimensional harmonic oscillator with Hamiltonian $H = p_x^2/2m + m\omega^2x^2/2$ when it is in its ground state. Here x is the position operator and p_x is the momentum operator.

B.5 Consider the 3-dimensional Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+V(\mathbf{r})\right]\psi(\mathbf{r},t) \quad .$$

For a given time-independent potential $V(\mathbf{r})$ the Hamiltonian H possesses a complete set of stationary solutions $\chi_j(\mathbf{r},t) = \chi_j(\mathbf{r})e^{-i\omega_j t}$ with energies $E_j = \hbar\omega_j$ where the eigenfunctions $\chi_j(\mathbf{r})$ form an orthonormal set. In terms of the Green's function $G(\mathbf{r}',t';\mathbf{r},t)$ any general wave function at time t' may related to that at time t according to

$$\psi(\mathbf{r}',t') = i\int d^3r\, G(\mathbf{r}',t';\mathbf{r},t)\,\psi(\mathbf{r},t) \quad . \qquad (1)$$

(a) For the above Schrödinger equation show that the function

$$G(\mathbf{r}',t';\mathbf{r},t) = -i\sum_j e^{-i\omega_j(t'-t)}\chi_j(\mathbf{r}')\chi_j^*(\mathbf{r}) \qquad (2)$$

will serve as the Green's function needed for Eq. (1).

(b) In terms of the above Green's function is convenient to define a *retarded* Green's function which only propagates forward in time (viz. $t' > t$) according to

$$G^+(\mathbf{r}',t';\mathbf{r},t) = \theta(t'-t)G(\mathbf{r}',t';\mathbf{r},t) \quad .$$

Here θ is the usual Heaviside unit step function: $\theta(\tau<0)=0$, $\theta(\tau>0)=1$ which can be represented as the $\epsilon \to 0^+$ limit of the integral:

$$\theta(\tau) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{d\omega e^{-i\omega\tau}}{\omega+i\epsilon}.$$

Show that the Fourier transform of the retarded Green's function which is defined via

$$G^+(\mathbf{r}',t';\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega G_\omega^+(\mathbf{r}';\mathbf{r})e^{-i\omega(t'-t)}$$

can be written as

$$G_\omega^+(\mathbf{r}';\mathbf{r}) = \sum_j \frac{\chi_j(\mathbf{r}')\chi_j^*(\mathbf{r})}{(\omega-\omega_j+i\epsilon)}$$

where ϵ is small and positive.

(c) Using the result of part (b), show that for a free particle case where $V(\mathbf{r})=0$, the retarded Green's function can be written in the closed form

$$G^+(\mathbf{r}',t';\mathbf{r},t) = -i\theta(t-t')\left[\frac{m}{2\pi i\hbar(t'-t)}\right]^{3/2}\exp\left[\frac{im|\mathbf{r}-\mathbf{r}'|^2}{2\hbar(t'-t)}\right] \quad .$$

B.6 The Dirac Hamiltonian for a relativistic spin one-half electron of mass m and charge e in an external electromagnetic potential $(\phi, \vec{A})$ is given by

$$H = e\phi + \vec{\alpha} \cdot (\vec{p} - e\vec{A}) + \beta m$$

where β, α_x, α_y and α_z the are four-dimensional Dirac matrices which obey

$$\alpha_x\alpha_y + \alpha_y\alpha_x = 0 \ , \ \alpha_y\alpha_z + \alpha_z\alpha_y = 0 \ , \ \alpha_z\alpha_x + \alpha_x\alpha_z = 0 \ , \ \alpha_x\beta + \beta\alpha_x = 0 \ , \ \alpha_y\beta + \beta\alpha_y = 0 \ , \ \alpha_z\beta + \beta\alpha_z = 0 \ ,$$

$$\alpha_x^2 = 1 \ , \ \alpha_y^2 = 1 \ , \ \alpha_z^2 = 1 \ , \ \beta^2 = 1$$

(a) Show that for this theory the time derivatives of the position and momentum operators $\vec{r}$ and $\vec{\pi}$ are given by

$$\frac{d\vec{r}}{dt} = \vec{\alpha} \ , \ \frac{d\vec{\pi}}{dt} = e(\vec{E} + \vec{\alpha} \times \vec{B})$$

where $\vec{\pi} = \vec{p} - e\vec{A}$ and $\vec{E}$ and $\vec{B}$ are the external electric and magnetic fields.

(b) Use these results to show that

$$\frac{d}{dt}\left(\vec{r} \times \vec{\pi} + \frac{\hbar}{2}\vec{\sigma}'\right) = \vec{r} \times \vec{F} \ , \ \frac{dM}{dt} = e\vec{\alpha} \cdot \vec{E}$$

where $M = \vec{\alpha} \cdot \vec{\pi} + \beta m$ and $\vec{F} = e(\vec{E} + \vec{\alpha} \times \vec{B})$, and where the matrix $\vec{\sigma}'$ is defined via

$$\alpha_i\alpha_j - \alpha_j\alpha_i = 2i\epsilon_{ijk}\sigma'_k$$

and has components

$$\sigma'_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \ \sigma'_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \ \sigma'_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$