Quantum Mechanics Prelim, Friday May 7, 2021. 8:00am-12:00pm

This exam is in two parts: **PART A** (QMI material) with **THREE** questions and **PART B** (QMII material) with **THREE** questions. Answer **TWO** questions form **PART A** and **TWO** questions from **PART B**. If you submit solutions to more than required number of solutions only the first two of your solutions to part A and the first two of your solutions to part B will be graded.

To take the prelims remotely students will need a good internet connection, a computer with a camera, a cell phone with a camera, and sufficient cell phone data capacity to switch to the data line on the cell phone if the wifi fails. Also students should keep their cell phone fully charged in case of power outages. Students should have the webex app on both their computer and cell phone. Each webex link will open at 7:45am. The computer camera will only be needed to check each student's ID prior to the start of the exam. The exam will be emailed to each participant at the starting time of the exam. Students should immediately download the exam to their computer and cell phone (and even print it out if they can) in case they lose the internet connection.

Students should write their solutions on blank 8.5 by 11 paper, putting their name on each page, the number of the problem and the number of the page in their solution (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their exams in sequence using the cell phone or a scanner (it might be easier to transfer the files to a laptop first) and email them in a file or files (ideally pdf) to philip.mannheim@uconn.edu, gayanath.fernando@uconn.edu and alexander.kovner@uconn.edu no later than 15 minutes after the end time of the exam, and the files will be checked to see that they are readable or if a resend is required. Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems.

During the exam students must keep the webex link live on both the computer and the cell phone, but only need to keep the cell phone camera on. Questions that arise should be asked through the chat on the computer webex, and students should arrange for at least the chat portion of the computer webex to be visible to them during the exam. Students can work on the same desk as they place their computer so that their hands are visible. The cell phone should be mounted (scotch tape on a hard vertical surface should suffice) so that the phone shows the computer screen and the entire work area. Proctors will monitor the students through the cell phone camera webex.

Some possibly useful information

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$$\boldsymbol{\nabla}\psi = \mathbf{e}_x\frac{\partial\psi}{\partial x} + \mathbf{e}_y\frac{\partial\psi}{\partial y} + \mathbf{e}_z\frac{\partial\psi}{\partial z} = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} = \mathbf{e}_\rho\frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z}.$$

$$\text{Hermite polynomial} = H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \quad , \quad H_0(x) = 1 \quad , \quad H_1(x) = 2x \quad , \quad H_2(x) = 4x^2 - 2$$

$$\text{Laguerre} = L_n(r) = e^r\frac{d^n}{dr^n}\left(r^n e^{-r}\right) \quad , \quad \text{associated Laguerre} = L^q_{n+q}(r) = (-1)^q\frac{d^q}{dr^q}L_{n+q}(r) \quad .$$

$$\text{Legendre polynomial} = P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l \quad , \quad P_0(x) = 1 \quad , \quad P_1(x) = x \quad , \quad P_2(x) = \frac{1}{2}(3x^2-1) \quad ,$$

$$\int_{-1}^{+1} dw P_\ell(w) P_{\ell'}(w) = \frac{2}{(2\ell+1)}\delta_{\ell\ell'}$$

$$\text{associated Legendre polynomial} = P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}}{dx^{|m|}}P_l(x)$$

$$\text{spherical harmonic} = Y_l^m(\theta,\ \phi) = (-1)^m \left[\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} P_l^m(\cos\theta) e^{im\phi} \quad ,$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2} \quad , \quad Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \quad , \quad Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$$

$$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta - 1) \ , \ Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi} \ , \ Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$$

$$\text{spherical Bessels :} \quad j_l(r) = (-1)^\ell r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\sin r}{r}\right) \quad , \quad n_l(r) = (-1)^{(\ell+1)} r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \left(\frac{\cos r}{r}\right) \quad ,$$

$$\text{with asymptotic behavior} \quad j_\ell(r) \to \frac{\cos(r - \ell\pi/2 - \pi/2)}{r} \quad , \quad n_\ell(r) \to \frac{\sin(r - \ell\pi/2 - \pi/2)}{r} \quad .$$

$$j_0(r) = \frac{\sin r}{r} \quad , \quad n_0(r) = -\frac{\cos r}{r} \quad , \quad j_1(r) = \frac{\sin r}{r^2} - \frac{\cos r}{r} \quad , \quad n_1(r) = -\frac{\cos r}{r^2} - \frac{\sin r}{r} \quad ,$$

$$j_2(r) = \frac{3\sin r}{r^3} - \frac{\sin r}{r} - \frac{3\cos r}{r^2} \quad , \quad n_2(r) = -\frac{3\cos r}{r^3} + \frac{\cos r}{r} - \frac{3\sin r}{r^2} \quad .$$

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta).$$

Some Gaussian integrals:

$$\int_0^\infty dx e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad \int_0^\infty dx x^2 e^{-ax^2} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}, \quad \int_0^\infty dx x^4 e^{-ax^2} = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}.$$

For Harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ the raising and lowering operators are defined as

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} x + i\left(\frac{1}{2\hbar m\omega}\right)^{1/2} p; \qquad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} x - i\left(\frac{1}{2\hbar m\omega}\right)^{1/2} p.$$

Kinetic term in a fixed angular momentum sector

$$\frac{\mathbf{p}^2}{2m}[R(r)Y_l^m(\theta,\phi)] = \left[-\frac{1}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right]R(r)\right]Y_l^m(\theta,\phi).$$

Angular momentum raising and lowering operators

$$j_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\ |j,m\pm 1\rangle.$$

PART A

1. a) A particle of mass m moves in a spherical potential well of radius a where

$$V(r) = -V_0 \text{ when } r < a$$

and

$$V(r) = 0 \text{ when } r > a$$

(with $V_0 > 0$).
Write down the corresponding Schrödinger equation for a wave function $\psi(r)$ for all r in a state with zero angular momentum. Introduce $u = r\psi$ and obtain differential equations for u in the regions $r < a$ and $r > a$ and find the least value of V_0 such that there is a bound state of zero energy and zero angular momentum.

b) Consider a (non-relativistic) particle moving in a cylindrically symmetrical potential $V(\rho)$ where $\rho^2 = x^2 + y^2$. What complete set of commuting observables would you use to specify the eigenstates of the system? Explain briefly.

2. Consider two, independent harmonic oscillators, described by the operators a_1, a_1^+ and a_2, a_2^+.

a) Write down the (total) number operator as well as the normalized eigenstates, $|n_1, n_2>$ in terms of the above operators (and any other necessary items) for this problem, defining your symbols carefully.

b) Define

$$J_x = \frac{1}{2}(a_1^+ a_2 + a_2^+ a_1); \quad J_y = \frac{1}{2i}(a_1^+ a_2 - a_2^+ a_1)$$

and

$$J_z = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2).$$

Show that $[J_i, J_j] = i\epsilon_{ijk} J_k$ where $i, j, k = x, y, z$. (Note the similarity with angular momentum operators.)

c) Check whether $|n_1, n_2>$ is an eigenstate of J_z by evaluating $J_z|n_1, n_2>$.

3. The radial Schrödinger equation for the electron-nucleus relative motion in a Hydrogenic atom can be written as $[\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2\mu r^2} - \frac{Ze^2}{r}]\, u(r) = E\, u(r)$, where the full wavefunction $\psi(\mathbf{r}) = r^{-1}\, u(r)\, Y_{lm}(\hat{\mathbf{r}})$. (The symbols used above have their usual meanings.)

(a) Consider the radial equation above. Identify the radial terms that dominate as $r \to 0$ and $r \to \infty$ and guess the form of the radial functions for a bound state in the limiting cases $r \to 0$ and $r \to \infty$. Verify that your guesses satisfy the appropriate differential equations.

(b) Sketch (separately) the radial solutions to a Hydrogenic atom for (i) $4s$ and (ii) $3d$ functions. Which one of the above radial functions dominates at large r?

(c) What is the average value $< r >$ in the ground state? Compare this to the value of r at which the probability density peaks.
Possibly useful information: The exact ground state wavefunction is

$$\left(\frac{Z^3}{\pi {a_0}^3}\right)^{1/2} \exp\left(-\frac{Zr}{a_0}\right)$$

and

$$\int_0^\infty dr\, r^3 \exp(-\lambda r) = \frac{6}{\lambda^4}.$$

PART B

4. Consider a particle in 3 dimensions in a rotationally symmetric harmonic oscillator potential

$$H = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}\kappa\mathbf{x}^2$$

where $\mathbf{x} = (x, y, z)$.

a). What is the degeneracy of the first excited level (energy $E = 5/2\,\omega$)? What are the wave functions of the states corresponding to this energy?

The particle is subject to a small perturbation

$$\Delta V = \alpha z x$$

b). Find the first order corrections to the energies of the states **on the first excited level** of the nonperturbed oscillator. Be careful: the level is degenerate, so you have to use degenerate perturbation theory!

5. A spin 1/2 particle is placed in magnetic field in the z direction. The Hamiltonian of the system is

$$H_z = -\mu B S_z.$$

At time $t < 0$ the particle is in the lowest energy state of the Hamiltonian (spin up). An additional weak magnetic field b in the x direction is quickly turned on at $t = 0$. This field is then turned off at time T so that the Hamiltonian of the system as a function of time is given by

$$H = -\mu B S_z - f(t)\mu b S_x$$

where $f(t)$ is a step function: $f(t) = 0$ for $t < 0$ and $t > T$, while $f(t) = 1$ for $0 < t < T$. Also assume $b \ll B$.

a). Using first order perturbation theory find the probability that the particle will be found with spin down at $t \to \infty$.

b). Now **without invoking the perturbation theory** solve the time evolution of the spin **exactly** between times 0 and T. Using this result calculate the same probability as in part a) and compare it to the result you got in part a) to leading order in the parameter b.

c). Without calculation explain how your answer will change if the perturbation is turned on and off very slowly, so that $f(t)$ changes from 0 to 1 (and later from 1 to 0) on time scale $\Delta t \gg 1/\mu B$.

6. Consider a particle of mass m constrained to move **on a two dimensional plane** x, y. The particle moves in a central potential, so that the 2D Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} + V(r)$$

The angular momentum operator in 2D is defined as

$$L = xp_y - yp_x$$

a) Show that a state $\Psi_l(\mathbf{r}) = f(r)e^{il\phi}$ (r and ϕ are polar coordinates on a plane) is an eigenstate of L. Find the appropriate eigenvalue.

b). The parity operator P in 2D is defined by its action on an arbitrary wave function as

$$P\psi(x, y) = \psi(x, -y)$$

Prove that

$$P^\dagger L P = -L.$$

c) Show that the Hamiltonian is invariant under parity, i.e. $P^\dagger H P = H$. Using the parity symmetry show that all eigenstates of H with nonvanishing l are at least doubly degenerate.