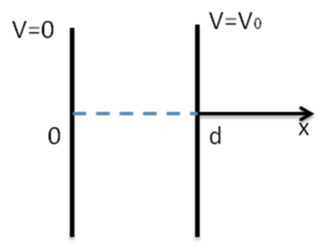
Preliminary Exam: Electromagnetism, Wednesday 26 August, 2020. 9:00-12:00

Answer a total of any **THREE** out of the four questions. If a student submits solutions to more than three problems, only the first three problems as listed on the exam will be graded. Students should write their solutions on blank 8.5 by 11 paper or in a blue book, putting their name on each page, the number of the problem and the number of the page in their solution on each page (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their solutions in sequence using a cell phone or a scanner and email them in a file or files to the prelim committee chair philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam. (It might be easier to transfer the files to a laptop first.) Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems. The chair will immediately check if the emailing is readable or if a resend is required.



1. Consider two plane-parallel electrodes, spaced by a distance d, at voltages 0 and V_0. The source that maintains the voltage difference V_0 between the two electrodes supplies whatever current is necessary to maintain that voltage difference as the charge is allowed to free-stream across the gap under the influence of the electric field.

   Assume (i) there is vacuum between the plates, (ii) the electrons are released from the inner surface of the plate with the lower voltage, (iii) ignore gravity, (iv) assume the area of plates is much greater than d^2, and (v) assume steady-state current flow.

   The velocity $v(x)$ of these electrons increases as a function of x. The potential between the plates, $V(x)$, is also a function x with $V(x = 0) = 0$ and $V(x = d) = V_0$.

   (**a**) Determine the Poisson equation satisfied by $V(x)$.

   (**b**) Write down the continuity equation that governs the current density in the gap as a function of x.

   (**c**) Solve this equation to determine the current density as a function of $V(x)$.

2. Two concentric metal spheres of radii a and b ($a < b$) are separated by a medium that has a dielectric constant ϵ and conductivity σ. The charge on the inner conductor q (for times $t < 0$) is constant q_0 because there is a thin insulating barrier between it and the dielectric. Then at $t = 0$ the insulating layer is suddenly removed, allowing charge to flow into the dielectric. Take the limit of small σ, so magnetic fields and inductive effects can be ignored. Assume the net charge density in the dielectric is everywhere zero.

   (**a**) Calculate the time dependence of the charge q.

   (**b**) Calculate the total current flowing through the medium at time t.

3. Electric charge is distributed non-uniformly over the surface of a hollow sphere of radius R. The surface charge density σ depends on the polar angle θ and azimuthal angle ϕ according to $\sigma = \sigma(\theta, \phi)$.

   (**a**) For the arbitrary $\sigma(\theta, \phi)$ derive an analytical formula for the electrostatic potential $\Phi(\mathbf{r})$ both inside and outside the spherical surface, and express it in terms of the spherical harmonics $Y_l^m(\theta, \phi)$.

   Hint: The spherical harmonics $Y_\ell^m(\theta, \phi)$ represent the complete set of orthonormal eigenfunctions of the angular part of Poisson's equation. The Green's function of Poisson's equation $G(\mathbf{r}, \mathbf{r}') = \mathbf{1}/|\mathbf{r} - \mathbf{r}'|$ can be expressed via the spherical harmonics as

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_\ell^{m*}(\theta', \phi') Y_\ell^m(\theta, \phi)$$

   where $r_<$ and $r_>$ are respectively the smaller and larger of $|\mathbf{r}|$ and $|\mathbf{r}'|$. The general spherical harmonic is

$$Y_\ell^m(\theta,\ \phi) \;=\; (-1)^m \left[\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}\right]^{1/2} P_\ell^m(\cos\theta) e^{im\phi} \qquad (1)$$

   A list of the first few spherical harmonics may be found at the end of this exam paper.

   (**b**) Determine the asymptotic multipole expansion of the potential $\Phi(\mathbf{r})$ at large distance $r \gg R$ if the surface charge density is cylindrically symmetric and given by the expression:

$$\sigma = \sigma(\theta) = \sigma_0 \left[\alpha + \beta\ \cos(\theta) + \gamma\ \cos^2(\theta)\right],$$

   where σ_0, α, β and γ are constants. Express the multipole moments via the given constants.

   (**c**) Find the combination of the α, β and γ parameters which makes the quadrupole term the leading term in the multipole expansion.

4. Consider a cylindrical coordinate system (s, ϕ, z) with the radius s and the azimuthal angle ϕ being in the $(x = s\cos\phi,\ y = s\sin\phi)$ plane. Consider an infinitely long cylindrical layer with inner and outer radii R_1 and R_2 with its central axis along the z axis, i.e. along $(x = 0,\ y = 0,\ z)$. Within the layer there is a cylindrically symmetric charge density $\rho(s)$. The charge density is zero outside the layer. The cylindrical layer rotates with a constant angular velocity ω about the central axis.

   (**a**) Derive the differential equation for the magnitude $A(s)$ of the vector potential $\mathbf{A}(s)$ in all three regions $s < R_1$, $R_1 < s < R_2$, $R_2 < s$ using Poisson's equation $\nabla^2\mathbf{A} = -\mu_0\mathbf{j}$, where $\mathbf{j}$ is the vector of electric current density.

   Hint: Chose the cylindrical gauge for the $\mathbf{A}(s)$ vector potential: $\mathbf{A}(s) = A(s)\hat{\mathbf{e}}_\phi$.

   (**b**) Using the equation obtained in part (**a**), determine the vector potential $\mathbf{A}(s)$ for the specific case of the cylindrically symmetric charge distribution in which the inner and outer layer surfaces are charged uniformly with charge densities σ and $-\sigma$ respectively: $\rho(s) = \sigma\delta(s - R_1) - \sigma\delta(s - R_2)$. Along the central axis take the vector potential to vanish, viz. $\mathbf{A}(s = 0, z) = 0$.

   Hint: Solve the equations for $\mathbf{A}(s)$ separately in the three regions, then write down the matching conditions at the boundaries of each region.

   (**c**) Calculate the total energy of magnetic field per unit of length in each of the three cylindrical regions: $s < R_1$, $R_1 < s < R_2$, and $s > R_2$.

SPHERICAL HARMONICS $Y_{lm}(\theta, \phi)$

$$l = 0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1 \quad \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}} \cos\theta \end{cases}$$

$$l = 2 \quad \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}} \sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\dfrac{5}{4\pi}} \left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) \end{cases}$$

$$l = 3 \quad \begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}} \sin^3\theta e^{3i\phi} \\ Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}} \sin^2\theta \cos\theta e^{2i\phi} \\ Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}} \sin\theta \left(5\cos^2\theta - 1\right) e^{i\phi} \\ Y_{30} = \sqrt{\dfrac{7}{4\pi}} \left(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta\right) \end{cases}$$

Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$