Preliminary Exam: Electromagnetism, Thursday August 22, 2019. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a **SEPARATE** blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. Consider a sphere of radius R centered at the origin. Suppose a point charge q is put at the origin and that is the only charge inside or outside the sphere. Furthermore, the potential is $\phi = V_0 \cos\theta$ on the surface of the sphere.

   (**a**) What is the electric potential inside the sphere?

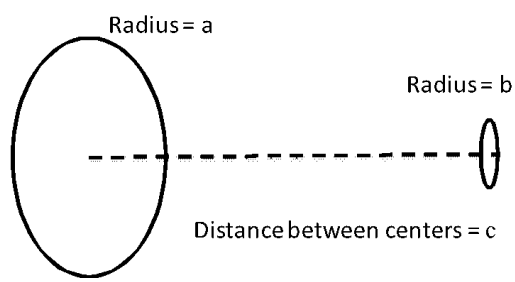
   (**b**) What is the electric field outside the sphere assuming the potential goes to zero as r approaches infinity?

2. Two single turn circular loops are mounted as shown in the figure, where $b \ll a$

   (**a**) Find the magnetic field at the center of the small loop created by the large loop carrying a current I.

   (**b**) Assuming the magnetic field at the center of the small loop equals the magnetic field at the wire of the small loop, find the mutual inductance between the loops. Note: the mutual inductance M between two loops A and B is the ratio between the magnetic flux ϕ_B that is produced in one loop (e.g. B) by the current I_A driving the other one (e.g. A), i.e. $M = \phi_B/I_A$

   (**c**) The self-inductance of each individual loop alone without the other is represented by $L_0(A)$ and $L_0(B)$, respectively. How is the self-inductance of the loop of radius a modified by the presence of loop of radius b? You should assume that current is free to flow around the loop of radius b without resistance.



3. A dielectric cylinder of radius R and length L carries an electric charge Q uniformly distributed over the volume of the cylinder. The cylinder is rotating about the cylindrical symmetry axis z with angular velocity $\vec{\omega} = \omega \hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ is a unit vector along the z-axis.

   (**a**) Find the magnetic moment $\mathbf{m}$ of the rotating cylinder, while neglecting bound charges induced in this perfect dielectric.

   (**b**) Determine the value of the magnetic moment for an identical cylinder, which carries the electric charge Q distributed uniformly only over the cylinder surface, including top and bottom bases.

   (**c**) Explain the difference in values of magnetic moments determined in parts **(a)** and **(b)**.

   (**d**) Calculate the magnetic field $\mathbf{B}$ and vector-potential $\mathbf{A}$ induced at a large distance r ($r \gg L, R$) by the rotating cylinders considered in parts **(a)** and **(b)**.

4. An infinite cloud of electrons is distributed above a grounded conducting plane lying in the horizontal (x, y) plane at $z = 0$. The volume charge density $\rho(z)$ of the distributed charge of electrons depends on the distance z from the infinite conducting plane at $z = 0$: $\rho(z) = -\rho_0 \exp(-z/a)$ for $z \geq 0$, where ρ_0 and a are positive constants and where the vertical z-axis is perpendicular to the horizontal conducting plane.

   (**a**) Solve Poisson's equation for the electric field potential $\phi(z)$ in the entire region $z \geq 0$ above the conducting surface. The reference surface for the electric field $\mathbf{E}(\mathbf{z})$ is chosen at $z \to \infty$: $\mathbf{E}(\mathbf{z}) = -\hat{\mathbf{e}}_z \frac{d}{dz}\phi(z \to \infty) \to 0$, where $\hat{\mathbf{e}}_z$ is a unit vector in the direction of the z-axis.

   (**b**) Calculate the electric field $\mathbf{E}(z)$ and find the surface charge density σ on the conducting plane.

   (**c**) Determine the electric dipole moment per unit of area of the system of the conducting plane and distributed electron cloud.

Vector Formulas

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla\psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla \cdot (\psi\mathbf{a}) = \mathbf{a} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi\mathbf{a}) = \nabla\psi \times \mathbf{a} + \psi\nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla \cdot \mathbf{x} = 3 \qquad \nabla \times \mathbf{x} = 0$$

$$\nabla \cdot [\mathbf{n}f(r)] = \frac{2}{r} f + \frac{\partial f}{\partial r} \qquad \nabla \times [\mathbf{n}f(r)] = 0$$

$$(\mathbf{a} \cdot \nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})] + \mathbf{n}(\mathbf{a} \cdot \mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a}) + i(\mathbf{L} \times \mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x} \times \nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right]$$
$$+ \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$