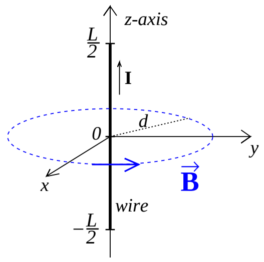
Preliminary Exam: Electromagnetism, Thursday August 27, 2015, 9:00-12:00

Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

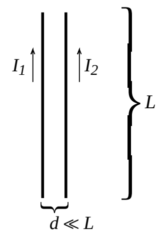
Remark: A possibly useful integral is $\displaystyle\int \frac{dz}{(a^2+z^2)^{3/2}} = \frac{z}{a^2(a^2+z^2)^{1/2}}$.

Problem 1

(a) Derive the Biot-Savart law from Maxwell's equations of magnetostatics. (Hint: Use the vector potential $\vec{A}(\vec{x})$, and choose the Coulomb gauge $\vec{\nabla} \cdot \vec{A}(\vec{x}) = 0$.)
(b) Apply the Biot-Savart law to a *finite* segment of a wire of length L carrying a current I to calculate its contribution to the magnetic field in the x-y-plane at a distance d from the center of the wire, as shown in the Figure.



for part (b)



for part (c)

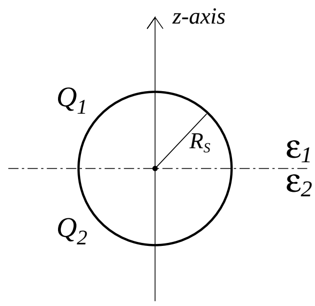
(c) Two wires of length L carry currents I_1 and I_2 and are separated by a distance d. Calculate the force per unit length between the wires (for $d \ll L$). Neglect edge effects.

Problem 2

(a) Find the resistance between two concentric spherical conductors of radii a, b ($a > b$) filled with a material of conductivity σ, dielectric constant ε, and permeability $\mu = 1$.
(b) Find the capacitance between two concentric spherical conductors of radii a, b ($a > b$) filled with an insulator with the dielectric constant ε, and permeability $\mu = 1$.

Problem 3

A conducting sphere with radius R_S carrying the charge $Q > 0$ is placed on the boundary of two dielectric media with $\varepsilon_1 \neq \varepsilon_2$. By solving the Laplace equation determine the electric potential $\phi(\vec{x})$ and the fields $\vec{E}(\vec{x})$ and $\vec{D}(\vec{x})$ for $|\vec{x}| > R_S$. Calculate how the total charge $Q = Q_1 + Q_2$ is partitioned between the regions of the sphere with $z > 0$ and $z < 0$.

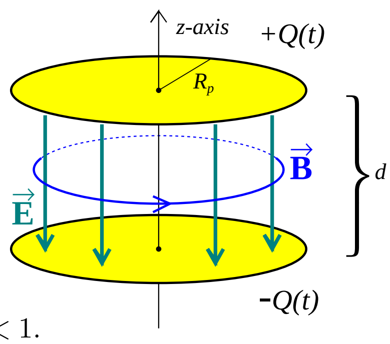


Hint: The general solution of the Laplace equation with azimuthal symmetry is given by

$$\phi(\vec{x}) = \sum_{l=0}^{\infty} (A_l\, r^l + B_l\, r^{-l-1})\, P_l(\cos\theta).$$

Problem 4

The charges $\pm Q(t)$ are uniformly distributed on circular plates with radius R_p of a parallel-plate capacitor, which is *slowly* discharged such that $Q(t) = Q_0 \exp(-t/\tau)$ where τ is a characteristic time such that the dimensionless quantity $x = R_p/(c\tau) \ll 1$.
(a) Calculate the electric field $\vec{E}(\vec{x}, t)$ and magnetic field $\vec{B}(\vec{x}, t)$ generated by the displacement current between the plates for $r \leq R_p$ where r denotes the distance from the z-axis.



(b) Show that your solution from part (a) is only valid for $x \ll 1$.
(c) Derive a result for the ratio $W_{\text{magnetic}}/W_{\text{electric}}$ of the energies contained in the magnetic and electric fields within the volume between the plates, and express the result in terms of x.

Remarks: Here "between the plates" means distances from the wire up to $r \leq R_p$. Use the simplifying assumption that the electric field is uniform between the plates for all $r \leq R_p$.