ELECTRICITY AND MAGNETISM

Preliminary Examination

August 22, 2013

9:00am - 12:00pm in P-121

Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on individual sheets of paper. Make sure you clearly indicate who you are and the problem you are solving on each book/sheet of paper. Double-check that you include everything you want graded, and nothing else.

On the last page you find the forms of the standard vector calculus operations in the three most common coordinate systems.

Problem 1. An electric charge is distributed with the volume charge density $\rho(x)$:

$$\rho(x) = \begin{cases} \rho_0 \ \exp(-x/a) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where ρ_0 and a are positive constants. The potential $\phi(\mathbf{r})$ and electric field $\mathbf{E}(\mathbf{r})$ are equal to zero in the plane $x = 0$: $\phi(\mathbf{r})|_{x=0} = 0$ and $\mathbf{E}(\mathbf{r})|_{x=0} = 0$. Calculate
(a) the electric potential $\phi(\mathbf{r})$ in the entire space;
(b) the potential energy of the ideal electric dipole $\mathbf{d} = d_0\mathbf{e}_x$ placed at an arbitrary point and oriented along the x-axis.

Problem 2. A very long conducting cylinder of radius R_0 is inserted into a constant (in space and time) electric field $\mathbf{E}_0$ so that the the cylinder axis is perpendicular to the direction of the electric field $\mathbf{E}_0$. Determine the new electric field established after the transients have died out.
Hint: In cylindrical coordinates ρ, ϕ, and z, any solution to the Laplace equation may be written as a linear combination of functions of the form $R(\rho)\Phi(\phi)Z(z)$. For the case of z-uniform solutions with $Z(z) = \text{const.}$, a basis $\{R_m(\rho)\Phi_m(\phi)\}$ is given by
1 and $\ln \rho$ for $m = 0$; and $\rho^m \cos m\phi$, $\rho^m \sin m\phi$, $\rho^{-m} \cos m\phi$, $\rho^{-m} \sin m\phi$ for $m = 1, 2, 3....$

Problem 3. An ideal dipole vector $\mathbf{d}$ (d=const. >0) is placed in front of an infinite planar conductor at the distance z_0 from the conductor surface. The z-axis is normal to the conductor surface. Determine the potential $\phi(\mathbf{r})$ of electric field in any arbitrary point and find the distribution of the induced surface charge $\sigma(x, y)$, for
(a) the dipole vector $\mathbf{d} = \mathrm{d}\ \mathbf{e}_z$;
(b) the dipole vector $\mathbf{d} = \mathrm{d}\ \mathbf{e}_y$ (the unit vector $\mathbf{e}_y$ is parallel to the conductor plane);
(c) the dipole vector $\mathbf{d} = \mathrm{d}\ (\mathbf{e}_y + \mathbf{e}_z)/\sqrt{2}$.

Problem 4. In a coaxial cable the electromagnetic field is confined between two conducting cylinders, call their radii a and b with $b > a$.
(a) Write Maxwell's equations and the appropriate boundary conditions describing the electromagnetic field in this coaxial cable.
(b) Show that the electromagnetic waves in the coaxial cable have a transverse electromagnetic (TEM) mode whose electric field is

$$\mathbf{E}(\mathbf{r}, t) = \frac{A}{\rho} e^{i(kz-\omega t)} \mathbf{e}_\rho, \tag{1}$$

where A is a constant characterizing the strength of the electric field, and k and ω are the wave number and the frequency, respectively.

Standard vector operations in three common coordinate systems

Cartesian coordinates x, y, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{e}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{e}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

cylindrical coordinates ρ, ϕ, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \hat{\mathbf{e}}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \hat{\mathbf{e}}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \\
\nabla^2 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

spherical polar coordinates r, θ, ϕ

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_r \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] + \hat{\mathbf{e}}_\theta \left[\frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} \right] + \hat{\mathbf{e}}_\phi \frac{1}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \\
\nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \\
&\left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} r \right]
\end{aligned}
$$