ELECTRICITY AND MAGNETISM

Preliminary Examination

Thursday August 23, 2012
09:00 - 12:00, P-121

Answer a total of THREE questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented separately in an answer book or on individual sheets of paper. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

On the last page you find the forms of the standard vector calculus operations in the three most common coordinate systems.

1. Consider a spherically symmetric charge distribution whose charge density is given by:

$$\rho(r) = A\,r \qquad \text{for } r < a$$

$$= B/r \qquad \text{for } a < r < b$$

$$= 0 \qquad \text{for } r > b$$

(a) Find the value of B (in terms of A, a and b) for which the electric field in the region $a < r < b$ has a constant magnitude.

For this value of B, calculate the energy stored in the electric field in the three regions of space:

(b) $r < a$

(c) $a < r < b$

(d) $r > b$.

2. One plate of a thin parallel plate capacitor of separation d is kept at the potential $\phi = 0$, the other at $\phi = V$. Between the capacitor plates is a space charge density $\rho = kx$, where k is a constant and x is the distance from the plate with $\phi = 0$.

(a) Solving Poisson's equation, find the potential distribution ϕ in the capacitor.

(b) Find the electric field E in the capacitor.

3. Suppose an electrically conducting grounded sphere of radius a is inserted in the initially uniform electric field $\vec{E} = E_o\,\hat{e}_z$. Calculate the potential ϕ at all points outside of the sphere. Hints and tools:

(a) Choose the polar axis of the sphere along the z axis.
(b) What is the form of the potential ϕ for z >> a?
(c) What is the potential at r = a?
(d) Solutions of Laplace's equation in spherical coordinates assuming azimuthal symmetry are:

$$\sum_{l=0}^{\infty}\left[A_l\, r^l + B_l\, r^{-(l+1)}\right]P_l(\cos\theta)$$

(e) An orthogonality relation:

$$\int_{-1}^{+1} P_{l'}(x)P_l(x)dx = \frac{2}{2l+1}\delta_{ll'}$$

(f) The first several Legendre polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2} - \frac{3}{2}x^2$$

4. A phenomenological model of a superconductor is given by the so-called London equations

$$c\nabla \times \lambda \vec{j} = -\vec{H} \qquad (\lambda \quad \text{constant})$$

$$\frac{\partial}{\partial t}(\lambda \vec{j}) = \vec{E}$$

rather than Ohm's law $\vec{j} = \sigma \vec{E}$. Consider a semi-infinite slab of the superconducting material occupying the region z > 0, outside of which is a constant magnetic field parallel to the surface, $H_x = H_z = 0$, $H_y = H_0$ for z ≤ 0, with $\vec{E} = \vec{D} = 0$ everywhere. Assume that no surface currents or surface charges are present.

(a) Using Maxwell's equations (in any system of units) and the London equations derive an equation for the magnetic field $\vec{H}$, and solve the equation to find the magnetic field inside the superconducting slab.

(b) Find the current density $\vec{j}$ inside the superconducting slab.

Standard vector operations in three common coordinate systems

Cartesian coordinates x, y, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{e}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{e}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

cylindrical coordinates ρ, ϕ, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \hat{\mathbf{e}}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \hat{\mathbf{e}}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \\
\nabla^2 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

spherical polar coordinates r, θ, ϕ

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_r \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] + \hat{\mathbf{e}}_\theta \left[\frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \right] + \hat{\mathbf{e}}_\phi \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \\
\nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \\
&\left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} r \right]
\end{aligned}
$$