Prelim Exam: Electrodynamics, Tuesday December 15, 2020. 6:00pm-9:00pm

Answer a total of any **THREE** out of the four questions. If a student submits solutions to more than three problems, only the first three problems as listed on the exam will be graded. Students should write their solutions on blank 8.5 by 11 paper or in a blue book, putting their name on each page, the number of the problem and the number of the page in their solution on each page (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their solutions in sequence using a cell phone or a scanner and email them in a file or files (ideally pdf) to the prelim committee chair philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam. (It might be easier to transfer the files to a laptop first.) Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems. The chair will immediately check if the emailing is readable or if a resend is required.

1. Electric charge is distributed inside a grounded sphere of radius R. The volume charge density $\rho(\mathbf{r})$ is cylindrically symmetric and depends on the value of the radius r and the polar angle θ between the radius-vector $\mathbf{r}$ and the axis of cylindrical symmetry z: $\rho = \rho(r, \theta)$.

   (a) For an arbitrary function $\rho(r, \theta)$, derive an analytical formula for the electrostatic potential $\Phi(\mathbf{r})$ inside the sphere and express it in terms of the Legendre polynomials $P_\ell(\cos\theta)$.
   Hint: The Green's function $G(\mathbf{r}, \mathbf{r}')$ inside the grounded sphere can be expressed via the spherical harmonics $Y_{\ell,m}(\theta, \phi)$:

   $$G(\mathbf{r}, \mathbf{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{m=\ell} \frac{4\pi}{2l+1}\, Y^*_{\ell,m}(\theta', \phi') Y_{\ell,m}(\theta, \phi)\, \left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r_<^\ell\, r_>^\ell}{R^{2\ell+1}}\right),$$

   where $r_<$ and $r_>$ are respectively the smaller and larger of $|\mathbf{r}|$ and $|\mathbf{r}'|$. For a cylindrically symmetric distribution with $m = 0$, all the spherical harmonics reduce to the Legendre polynomials: $Y_{\ell,m=0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$.

   (b) Determine the electric potential $\Phi(\mathbf{r})$ inside the sphere in the case of a uniform line of charge with linear charge density λ along the z-axis from $-R$ to R.
   Hint: $P_\ell(x = 1) = 1$ and $P_\ell(x = -1) = (-1)^\ell$.

2. A non-uniform electric current propagates along the x-axis in the upper part of a space with $z \geq 0$. The volume density of electric current $\mathbf{j}(\mathbf{r})$ depends on the z-coordinate: $\mathbf{j}(\mathbf{r}) = j_0 \exp(-z/d)\ \hat{\mathbf{e}}_x$, where j_0 and d are positive constants and $\hat{\mathbf{e}}_x$ is a unit vector in the x direction. A surface current with surface current density K propagates along the plane $z = 0$ in the direction -$\hat{\mathbf{e}}_x$.

   (a) Calculate the value K of the surface current that makes the magnetic field vanish at infinity: $B(z = \pm\infty) = 0$.

   (b) Determine the vector potential $\mathbf{A}(\mathbf{r})$ of the magnetic field for the K-value calculated in part (a) if the vector potential is zero on the plane $z = 0$.
   Hint: If you work in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ then this problem reduces to a one-dimensional differential equation that you can solve subject to the stated boundary conditions.

   (c) Calculate the energy of the magnetic field inside an infinite cylinder with cross section S and symmetry axis along the z axis.

3. Consider two infinite planes each one with a fixed value of x, and with both the y and z coordinates of the planes running from $-\infty$ to ∞. The two planes are parallel to each other, one located at $x = -a$ and the other at $x = a$. The space between the two planes is filled with a dielectric material. Embedded in the dielectric is free electric charge of a fixed volume density $\rho_f(x)$ that only depends on x. The dielectric constant of the medium also depends on x alone as $\epsilon(x)$.

   (a) If the electric potential $\phi(x)$ and electric field $E(x)$ of the system have zero values at $x = -a$ show that the electric potential $\phi(x)$ at any given x between the plates is given by the formula:

$$\phi(x) = -\int_{-a}^{x} \frac{dx'}{\epsilon(x')} \int_{-a}^{x'} \rho_f(x'')dx'', \qquad |x| \leq a.$$

   (b) For the special case of

$$\rho_f(x) = \rho_0 = \text{constant}, \quad \epsilon(x) = \epsilon_0(1 + x^2/a^2)$$

   calculate the energy W stored in the electric field within a cylinder of cross sectional area S whose axis is the x axis and whose end caps are at $x = -a$ and $x = a$.

   (c) Evaluate the vector of polarization $\mathbf{P}(x)$ and the volume density of bound charges $\rho_b(x)$ in the dielectric with the parameters defined in part (b).

4. A straight and infinitely long strip of width b carries a uniformly distributed electric current I flowing in the direction of the z-axis. Axes x and z lie in the strip plane and the y-axis is perpendicular to the strip plane.

   (a) Calculate the vector of magnetic induction $\mathbf{B}(\mathbf{r})$ in the entire space.
   Hint: It is convenient to take the origin of coordinates (x, y, z) at the center of the strip and consider a superposition of magnetic fields induced by infinitesimal straight currents.

   (b) Determine the asymptotic behavior of the magnetic field $\mathbf{B}(\mathbf{r})$ at large distances from the strip, where $s = \sqrt{x^2 + y^2} \gg b$.

   (c) Find the energy of interaction between a point magnetic dipole moment $\mathbf{m}$ and this strip, if the dipole moment is oriented along the y axis and $s \gg b$.

Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A} \, d^3x = \int_S \mathbf{A} \cdot \mathbf{n} \, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla \psi \, d^3x = \int_S \psi \mathbf{n} \, da$$

$$\int_V \nabla \times \mathbf{A} \, d^3x = \int_S \mathbf{n} \times \mathbf{A} \, da$$

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, d^3x = \int_S \phi \mathbf{n} \cdot \nabla \psi \, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} \, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla \psi \, da = \oint_C \psi \, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$