1. Electric charge is distributed inside a grounded sphere of radius $R$. The volume charge density $\rho(r)$ is cylindrically symmetric and depends on the value of the radius $r$ and the polar angle $\theta$ between the radius-vector $r$ and the axis of cylindrical symmetry $z$: $\rho = \rho(r, \theta)$.

(a) For an arbitrary function $\rho(r, \theta)$, derive an analytical formula for the electrostatic potential $\Phi(r)$ inside the sphere and express it in terms of the Legendre polynomials $P_{\ell}(\cos \theta)$.

**Hint:** The Green’s function $G(r, r')$ inside the grounded sphere can be expressed via the spherical harmonics $Y_{\ell,m}(\theta, \phi)$:

$$G(r, r') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell + 1} Y_{\ell,m}^*(\theta', \phi') Y_{\ell,m}(\theta, \phi) \left( \frac{r_<}{r_{\ell+1}^{\ell}} - \frac{r_>^{\ell}}{R^{2\ell+1}} \right),$$

where $r_<$ and $r_>$ are respectively the smaller and larger of $|r|$ and $|r'|$. For a cylindrically symmetric distribution with $m = 0$, all the spherical harmonics reduce to the Legendre polynomials: $Y_{\ell,m=0}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_{\ell}(\cos \theta)$.

(b) Determine the electric potential $\Phi(r)$ inside the sphere in the case of a uniform line of charge with linear charge density $\lambda$ along the $z$-axis from $-R$ to $R$.

**Hint:** $P_{\ell}(x = 1) = 1$ and $P_{\ell}(x = -1) = (-1)^{\ell}$.

2. A non-uniform electric current propagates along the $x$-axis in the upper part of a space with $z \geq 0$. The volume density of electric current $j(r)$ depends on the $z$-coordinate: $j(r) = j_0 \exp(-z/d) \hat{e}_x$, where $j_0$ and $d$ are positive constants and $\hat{e}_x$ is a unit vector in the $x$ direction. A surface current with surface current density $K$ propagates along the plane $z = 0$ in the direction $-\hat{e}_x$.

(a) Calculate the value $K$ of the surface current that makes the magnetic field vanish at infinity: $B(z = \pm \infty) = 0$.

(b) Determine the vector potential $A(r)$ of the magnetic field for the $K$-value calculated in part (a) if the vector potential is zero on the plane $z = 0$.

**Hint:** If you work in the Coulomb gauge $\nabla \cdot A = 0$ then this problem reduces to a one-dimensional differential equation that you can solve subject to the stated boundary conditions.

(c) Calculate the energy of the magnetic field inside an infinite cylinder with cross section $S$ and symmetry axis along the $z$ axis.
3. Consider two infinite planes each one with a fixed value of $x$, and with both the $y$ and $z$ coordinates of the planes running from $-\infty$ to $\infty$. The two planes are parallel to each other, one located at $x = -a$ and the other at $x = a$. The space between the two planes is filled with a dielectric material. Embedded in the dielectric is free electric charge of a fixed volume density $\rho_f(x)$ that only depends on $x$. The dielectric constant of the medium also depends on $x$ alone as $\epsilon(x)$.

(a) If the electric potential $\phi(x)$ and electric field $E(x)$ of the system have zero values at $x = -a$ show that the electric potential $\phi(x)$ at any given $x$ between the plates is given by the formula:

$$
\phi(x) = -\int_{-a}^{x} \frac{dx'}{\epsilon(x')} \int_{-a}^{x'} \rho_f(x'')dx'', \quad |x| \leq a.
$$

(b) For the special case of

$$
\rho_f(x) = \rho_0 = \text{constant}, \quad \epsilon(x) = \epsilon_0(1 + x^2/a^2)
$$

calculate the energy $W$ stored in the electric field within a cylinder of cross sectional area $S$ whose axis is the $x$ axis and whose end caps are at $x = -a$ and $x = a$.

(c) Evaluate the vector of polarization $\mathbf{P}(x)$ and the volume density of bound charges $\rho_b(x)$ in the dielectric with the parameters defined in part (b).

4. A straight and infinitely long strip of width $b$ carries a uniformly distributed electric current $I$ flowing in the direction of the $z$-axis. Axes $x$ and $z$ lie in the strip plane and the $y$-axis is perpendicular to the strip plane.

(a) Calculate the vector of magnetic induction $\mathbf{B}(\mathbf{r})$ in the entire space.

\textbf{Hint:} It is convenient to take the origin of coordinates $(x, y, z)$ at the center of the strip and consider a superposition of magnetic fields induced by infinitesimal straight currents.

(b) Determine the asymptotic behavior of the magnetic field $\mathbf{B}(\mathbf{r})$ at large distances from the strip, where $s = \sqrt{x^2 + y^2} \gg b$.

(c) Find the energy of interaction between a point magnetic dipole moment $\mathbf{m}$ and this strip, if the dipole moment is oriented along the $y$ axis and $s \gg b$. 

Vector Formulas

\[ a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) \]
\[ a \times (b \times c) = (a \cdot c)b - (a \cdot b)c \]
\[ (a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \]
\[ \nabla \times \nabla \psi = 0 \]
\[ \nabla \cdot (\nabla \times a) = 0 \]
\[ \nabla \times (\nabla \times a) = \nabla(\nabla \cdot a) - \nabla^2 a \]
\[ \nabla \cdot (\psi a) = a \cdot \nabla \psi + \psi \nabla \cdot a \]
\[ \nabla \times (\psi a) = \nabla \psi \times a + \psi \nabla \times a \]
\[ \nabla(a \cdot b) = (a \cdot \nabla)b + (b \cdot \nabla)a + a \times (\nabla \times b) + b \times (\nabla \times a) \]
\[ \nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b) \]
\[ \nabla \times (a \times b) = a(\nabla \cdot b) - b(\nabla \cdot a) + (b \cdot \nabla)a - (a \cdot \nabla)b \]

If \( x \) is the coordinate of a point with respect to some origin, with magnitude \( r = |x| \), \( n = x/r \) is a unit radial vector, and \( f(r) \) is a well-behaved function of \( r \), then

\[ \nabla \cdot x = 3 \quad \nabla \times x = 0 \]
\[ \nabla \cdot [n f(r)] = \frac{2}{r} f + \frac{\partial f}{\partial r} \quad \nabla \times [n f(r)] = 0 \]
\[ (a \cdot \nabla) f(r) = \frac{f(r)}{r} [a - n(a \cdot n)] + n(a \cdot n) \frac{\partial f}{\partial r} \]
\[ \nabla (x \cdot a) = a + x(\nabla \cdot a) + i(L \times a) \]

where \( L = \frac{1}{i} (x \times \nabla) \) is the angular-momentum operator.
Theorems from Vector Calculus

In the following $\phi$, $\psi$, and $\mathbf{A}$ are well-behaved scalar or vector functions, $V$ is a three-dimensional volume with volume element $d^3x$, $S$ is a closed two-dimensional surface bounding $V$, with area element $da$ and unit outward normal $\mathbf{n}$ at $da$.

\[ \int_V \nabla \cdot \mathbf{A} \, d^3x = \int_S \mathbf{A} \cdot \mathbf{n} \, da \]  
(Divergence theorem)

\[ \int_V \nabla \psi \, d^3x = \int_S \psi \mathbf{n} \, da \]

\[ \int_V \nabla \times \mathbf{A} \, d^3x = \int_S \mathbf{n} \times \mathbf{A} \, da \]

\[ \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} \, da \]  
(Green's first identity)

\[ \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d^3x = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} \, da \]  
(Green's theorem)

In the following $S$ is an open surface and $C$ is the contour bounding it, with line element $dl$. The normal $\mathbf{n}$ to $S$ is defined by the right-hand-screw rule in relation to the sense of the line integral around $C$.

\[ \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da = \oint_C \mathbf{A} \cdot dl \]  
(Stokes's theorem)

\[ \int_S \mathbf{n} \times \nabla \psi \, da = \oint_C \psi \, dl \]
Explicit Forms of Vector Operations

Let \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and \( A_1, A_2, A_3 \) be the corresponding components of \( \mathbf{A} \). Then

\[
\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial \psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial \psi}{\partial x_3}
\]

\[
\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}
\]

\[
\nabla \times \mathbf{A} = \mathbf{e}_1 \left( \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \right) + \mathbf{e}_3 \left( \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \right)
\]

\[
\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2}
\]

Cylindrical

\( (\rho, \phi, z) \)

\[
\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial \rho} + \mathbf{e}_2 \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} + \mathbf{e}_3 \frac{\partial \psi}{\partial z}
\]

\[
\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_1) + \frac{1}{\rho} \frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}
\]

\[
\nabla \times \mathbf{A} = \mathbf{e}_1 \left( \frac{1}{\rho} \frac{\partial A_3}{\partial \phi} - \frac{\partial A_2}{\partial z} \right) + \mathbf{e}_2 \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho} \right) + \mathbf{e}_3 \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho A_2) - \frac{\partial A_1}{\partial \phi} \right)
\]

\[
\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}
\]

Spherical

\( (r, \theta, \phi) \)

\[
\nabla \psi = \mathbf{e}_1 \frac{1}{r} \frac{\partial \psi}{\partial r} + \mathbf{e}_2 \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \theta} + \mathbf{e}_3 \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi}
\]

\[
\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_1) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_2) + \frac{1}{r \sin \theta} \frac{\partial A_3}{\partial \phi}
\]

\[
\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta A_3) - \frac{\partial A_2}{\partial \phi} \right] + \mathbf{e}_2 \left[ \frac{1}{r \sin \theta} \frac{\partial A_1}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (r A_3) \right] + \mathbf{e}_3 \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_2) - \frac{\partial A_1}{\partial \theta} \right]
\]

\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}
\]

Note that \( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) = \frac{1}{r} \frac{\partial^2 \psi}{\partial r^2} (r \psi) \).