Preliminary Exam: Electromagnetism, Wednesday 15 January, 2020. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a **SEPARATE** blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. An infinitely long dielectric cylinder of radius R carries a non-uniform volume density of free charges: $\rho_f = \rho_0 \sin(\alpha r)$ if $r \leq R$, and $\rho_f = 0$ if $r > R$, where r is the radius in the cylindrical coordinate system, and ρ_0 and α are positive constants. The dielectric constant of the cylinder material is ϵ_r.

   (**a**) Calculate the potential $\phi(r)$ of the electric field in the entire space, if the potential reference surface is the boundary surface of the dielectric cylinder: $\phi(r = R) = 0$.

   Hint: The $\phi(r)$ potential inside the cylindrical region $r \leq R$ can be expressed via the sine integral function $\mathrm{Si}[x] = \int_0^x dt\, \frac{\sin(t)}{t}$.

   (**b**) Determine the polarization vector $\mathbf{P}$ as a function of the distance r and calculate the surface σ_b and volume ρ_b densities of induced bound charges.

   (**c**) Determine the value of the linear charge density λ that should be distributed uniformly along the cylinder axis Z to provide the zero-potential condition: $\phi(r) = 0$ for all $r \geq R$. What contribution do the induced (bound) charges in the dielectric make to your answer?

2. A uniform magnetic field $\mathbf{B}$ occupies a cylindrical region of radius R. It is of the form $\mathbf{B}(\mathbf{r}, z) = \mathbf{e}_z B_0$ for $r \leq R$, and $\mathbf{B}(\mathbf{r}, z) = 0$ for $r > R$, where $\mathbf{e}_z$ is a unit vector along the cylindrical axis Z $(-\infty < z < \infty)$, r is the radius in cylindrical coordinates, and B_0 is a positive constant.

   (**a**) Determine the vector potential $\mathbf{A}(r)$ in the entire space and present your answer using cylindrical coordinates, if $\mathbf{A}(r = 0) = 0$.

   (**b**) Find the density $\mathbf{j}(r)$ of the electric current, which generates this magnetic field.

   (**c**) Calculate the electric field $\mathbf{E}(r, t)$ induced by the $\mathbf{B}(\mathbf{r}, z)$ magnetic field if it decreases exponentially from time $t = 0$: $\mathbf{B}(\mathbf{r}, z) = \mathbf{e}_z B_0 \exp(-\gamma t)$, where γ is a positive constant.

   (**d**) How much energy per unit length is stored in the magnetic and electric fields inside the cylinder?

3. The potential of a spherical shell of radius R and of negligible thickness depends only on the polar angle θ and is given as $V(\theta)$. The space inside and outside the shell is vacuum.

   (**a**) Determine the potential $V(r, \theta)$ both inside $(r < R)$ and outside $(r > R)$ the shell.

   (**b**) Calculate the charge distribution on the spherical shell.

   (**c**) For the above find the potentials for $r < R$ and $r > R$ and the charge distribution for the case $V = V_0 \cos^2 \theta$, where V_0 is a constant.

   Hint: Try expanding V inside the shell in a power series in r, likewise in the region outside the shell, and then evaluate the coefficients in each expansion in terms of what you know V should be on the boundary between the two regions. Note, the first three Legendre polynomials are given by: $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = (3x^2 - 1)/2$.

4. As shown in the figure, a plane wave of frequency ω traveling in an infinite medium of refractive index n_0 is normally incident on another medium of index n_1. The second medium is of thickness d and has empty space of index $n_2 = 1$ beyond it.

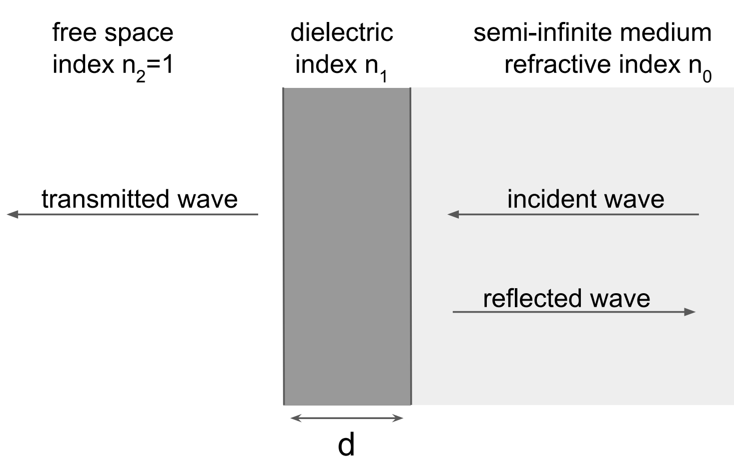
   (**a**) Write down plane wave solutions of frequency ω for the electric and magnetic fields in the three regions with refractive indices n_0, n_1 and n_2 respectively.

   (**b**) What are the boundary conditions that must be met at the two boundaries?

   (**c**) Find the amplitude of the reflected wave in terms of the incident electric field amplitude E_0.

   (**d**) Find the values of d for which the intensity in reflected wave is minimum.

   (**e**) Find the relationship between n_0 and n_1 for which there is no reflected wave, i.e. for which the minimum value in part (**d**) is zero.



Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial \psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial \psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial \rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial \psi}{\partial \phi} + \mathbf{e}_3 \frac{\partial \psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial \phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_2) - \frac{\partial A_1}{\partial \phi}\right)$$

$$\nabla^2 \psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla \psi = \mathbf{e}_1 \frac{\partial \psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \mathbf{e}_3 \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_2) + \frac{1}{r \sin\theta}\frac{\partial A_3}{\partial \phi}$$

$$\begin{aligned}\nabla \times \mathbf{A} = {} & \mathbf{e}_1 \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial \phi}\right] \\ & + \mathbf{e}_2\left[\frac{1}{r \sin\theta}\frac{\partial A_1}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(r A_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_2) - \frac{\partial A_1}{\partial \theta}\right]\end{aligned}$$

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$