ELECTRICITY AND MAGNETISM

Preliminary Examination

January 17, 2013

9:00 - 12:00 in P-121

Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on individual sheets of paper stapled together. Make sure you clearly indicate who you are, and the problem you are answering. Double-check that you include everything you want graded, and nothing else. On the last page you will find some "potentially useful formulas."

1. A point dipole $\vec{\mathbf{p}}$ is placed at the center of a sphere of linear dielectric material with the dipole axis pointing along the z-axis of a Cartesian coordinate system (see Fig. 1). The sphere has radius R and dielectric constant ϵ.

(a) Let us suppose that R is very large (i.e., $R \to \infty$). Write down an expression for the electrostatic potential at any point on the z-axis due to this dipole $\vec{\mathbf{p}}$.

(b) Now for finite $R > 0$, show that the electric potential inside the sphere is

$$\frac{p\cos\theta}{4\pi\epsilon_0\epsilon r^2}\left[1 + 2\frac{r^3(\epsilon - 1)}{R^3(\epsilon + 2)}\right]$$

where θ is the polar angle with respect to the z-axis. Compare and comment on your answer for part (a) with this one.

(c) Find the electric field at $r \gg R$. How does it behave in the limit $r \to \infty$?

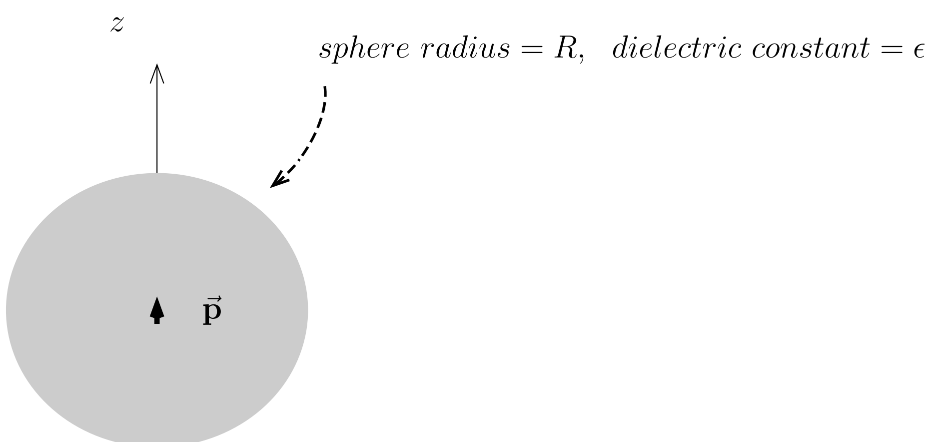

Fig. 1

2. A sphere of radius R carries a charge q that is distributed uniformly over the surface of the sphere. The sphere spins at constant angular velocity $\vec{\omega}$ (see Fig. 2).

(a) Calculate the magnetic moment $\vec{\mathbf{m}}$ of this sphere and determine the leading term of the multipole expansion of the vector potential $\vec{\mathbf{A}}$ at large distances r : i.e., $r \gg R$.

(b) Calculate the interaction energy between this sphere and an infinite straight wire carrying a steady current I. The angle between the direction of the current and the angular velocity vector $\vec{\omega}$ is θ while the distance ρ between the wire and the sphere is much greater than the sphere radius: i.e., $\rho \gg R$. Consider the case when $\vec{\omega}$ is perpendicular to $\hat{\mathbf{e}}_\rho$.

(c) Determine the interaction energy between the wire and the sphere, if the charge q is uniformly distributed over the sphere volume instead of the sphere surface.

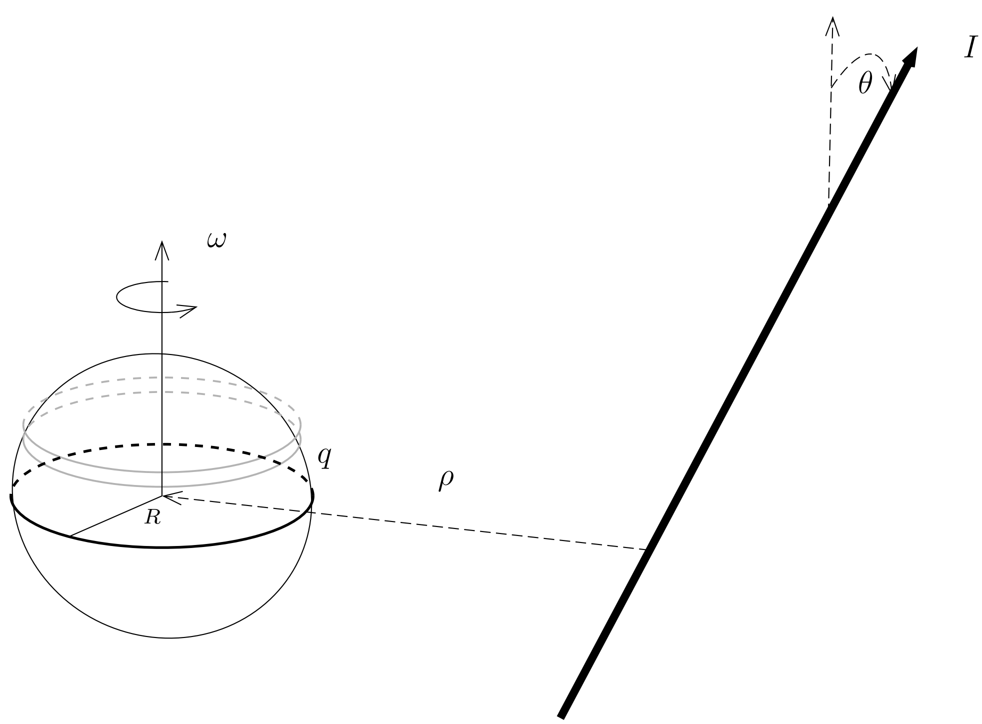

Fig. 2

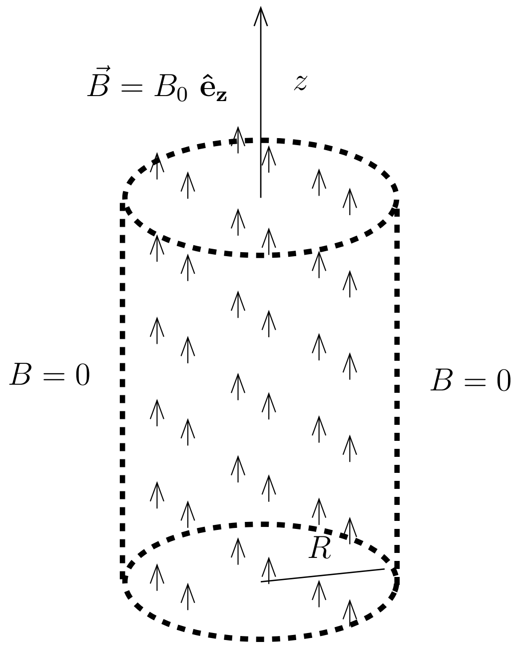

Fig. 3

3. Consider a magnetic field that has a constant magnitude and direction inside an <u>infinite cylinder</u> of radius R and vanishes everywhere outside (see Fig. 3): i.e.,

$$\vec{B}(\rho, \phi, z) = B_0\ \hat{\mathbf{e}}_z, \quad \rho \leq R \ \text{ and } \ B = |\vec{B}| = 0, \quad \rho > R$$

where $\hat{\mathbf{e}}_z$ is the unit vector along the axis of the cylinder and B_0 is a positive constant.

(a) Calculate the vector potential $\vec{A}(\rho, \phi, z)$ in the entire space.

(b) There is a current density associated with the above magnetic field as dictated by Maxwell's equations. Find the density of the electric current that induces this magnetic field.

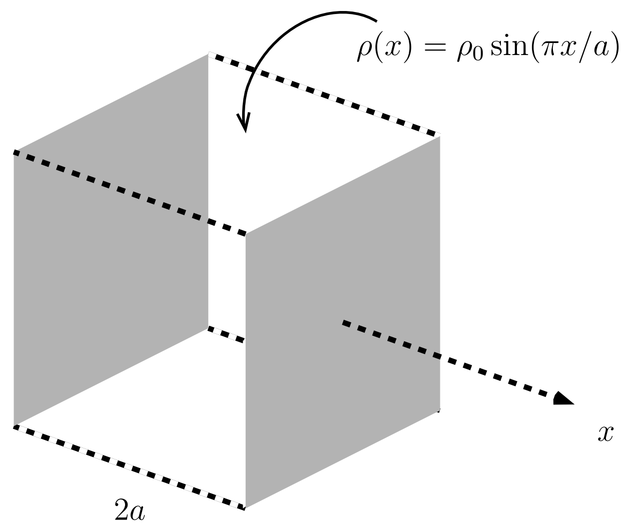

Fig. 4

4. An infinite slab of thickness $2a$ has a volume charge density $\rho(x)$ given by

$$\rho(x) = \begin{cases} \rho_0 \sin(\pi x/a) & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$$

where ρ_0 and a are positive constants (see Fig. 4). The geometry of this system is such that $x = 0$ is the central plane contained inside the slab with the $x-$axis being perpendicular to it. In addition, take this plane to be the potential reference plane; i.e., $\phi(x = 0) = 0$. For the above charge distribution, calculate the potential $\phi(x)$ and the electric field $E(x)$ everywhere in space.

Standard vector operations in three common coordinate systems

Cartesian coordinates x, y, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{e}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{e}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

cylindrical coordinates ρ, ϕ, z

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \\
\nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_\rho \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \hat{\mathbf{e}}_\phi \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] + \hat{\mathbf{e}}_z \frac{1}{\rho} \left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \\
\nabla^2 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}
$$

spherical polar coordinates r, θ, ϕ

$$
\begin{aligned}
\nabla &= \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \\
\nabla \cdot \mathbf{A} &= \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \\
\nabla \times \mathbf{A} &= \hat{\mathbf{e}}_r \frac{1}{r \sin\theta} \left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] + \hat{\mathbf{e}}_\theta \left[\frac{1}{r \sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi) \right] + \hat{\mathbf{e}}_\phi \frac{1}{r} \left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \\
\nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \\
& \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \equiv \frac{1}{r} \frac{\partial^2}{\partial r^2} r \right]
\end{aligned}
$$