Course and Prelim Exam: Electrodynamics, Tuesday May 4, 2021. 8:00am-11:00am

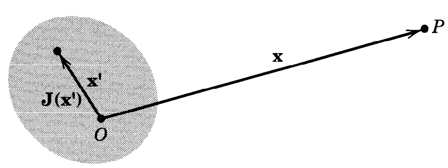
Answer a total of any **THREE** out of the four questions.

To take the prelims remotely students will need a good internet connection, a computer with a camera, a cell phone with a camera, and sufficient cell phone data capacity to switch to the data line on the cell phone if the wifi fails. Also students should keep their cell phone fully charged in case of power outages. Students should have the webex app on both their computer and cell phone. Each webex link will open at 7:45am. The computer camera will only be needed to check each student's ID prior to the start of the exam. The exam will be emailed to each participant at the starting time of the exam. Students should immediately download the exam to their computer and cell phone (and even print it out if they can) in case they lose the internet connection.

Students should write their solutions on blank 8.5 by 11 paper, putting their name on each page, the number of the problem and the number of the page in their solution (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their exams in sequence using the cell phone or a scanner (it might be easier to transfer the files to a laptop first) and email them in a file or files (ideally pdf) to philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam, and the files will be checked to see that they are readable or if a resend is required. Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems.

During the exam students must keep the webex link live on both the computer and the cell phone, but only need to keep the cell phone camera on. Questions that arise should be asked through the chat on the computer webex, and students should arrange for at least the chat portion of the computer webex to be visible to them during the exam. Students can work on the same desk as they place their computer so that their hands are visible. The cell phone should be mounted (scotch tape on a hard vertical surface should suffice) so that the phone shows the computer screen and the entire work area. Proctors will monitor the students through the cell phone camera webex.

1. A localized current $\mathbf{J}(\mathbf{x}')$ is confined to a region of volume V and arbitrary shape with a maximum value for $|\mathbf{x}'| = a$. It gives rise to a magnetic field $\mathbf{B}(\mathbf{x})$ at points P with $|\mathbf{x}| > a$.



   (a) In terms of the magnetic moment $\mathbf{m}$

$$\mathbf{m} = \frac{1}{2}\int d^3x'\mathbf{x}' \times \mathbf{J}(\mathbf{x}')$$

   as integrated over the the localized region V, show that at distances $|\mathbf{x}| \gg a$ the vector potential that is produced is of the form

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3}$$

   Hint: You may find it helpful to work in the gauge in which $\mathbf{A} \cdot \mathbf{x} = 0$ at large $|\mathbf{x}| \gg a$.

   (b) Determine the magnetic field $\mathbf{B}(\mathbf{x})$ at distances $|\mathbf{x}| \gg a$.

2. $Y_{\ell m}(\theta,\phi)$ with integer ℓ and m are spherical harmonics that obey

$$-\left[\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y_{\ell m}(\theta,\phi)=\ell(\ell+1)Y_{\ell m}(\theta,\phi),$$

$$\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta Y^*_{\ell' m'}(\theta,\phi)Y_{\ell m}(\theta,\phi)=\delta_{\ell\ell'}\delta_{mm'},$$

and are given in terms of associated Legendre functions $P_\ell^m(x)$ and Legendre functions $P_\ell(x)$ as

$$Y_{\ell m}(\theta,\phi)=\left(\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}\right)^{1/2}P_\ell^m(\cos\theta)e^{im\phi},$$

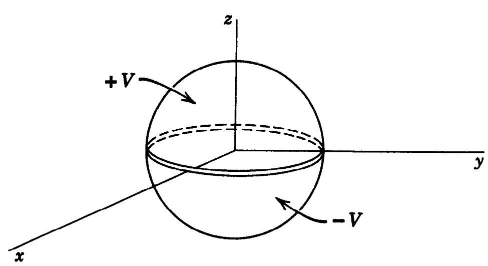
$$P_\ell^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x),\quad P_\ell(x)=\frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell.$$

A list of the first few spherical harmonics may be found at the end of this exam paper.

In terms of a general $f_{\ell,m}(r)$ an arbitrary potential function $\Phi(r,\theta,\phi)$ can be written in the form

$$\Phi(r,\theta,\phi)=\sum_{\ell=0}^{\ell=\infty}\sum_{m=-\ell}^{m=\ell}f_{\ell,m}(r)Y_{\ell m}(\theta,\phi).$$

(a) For an electrostatic potential $\Phi(r,\theta,\phi)$ that obeys the Laplace equation $\nabla^2\Phi=0$ determine the most general form for the coefficients $f_{\ell,m}(r)$.



(b) Consider a conducting sphere of radius a that is broken up into two hemispheres separated by an insulating ring of radius a lying in the $z=0$ plane. A constant potential V is put on the upper hemisphere $(0<\theta<\pi/2)$ and a constant potential $-V$ is put on the lower hemisphere $(\pi/2<\theta<\pi)$. Using the results from part (a) determine the potential at all points outside the sphere and give the coefficients of the $\ell=0$, $\ell=1$, and $\ell=2$ terms in an exact form in which all integrals have been evaluated.

3. (a) In a continuous distribution of charge and current in a volume V the total rate of doing work by the electric and magnetic fields is given by $\int d^3x\mathbf{J}\cdot\mathbf{E}$ as integrated over V. In this region the fields obey $\mathbf{D}=\epsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$, where ϵ and μ are constants. The energy density u and Poynting vector $\mathbf{S}$ for the electric and magnetic fields are of the form

$$u=\tfrac{1}{2}\left(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H}\right),\quad \mathbf{S}=\mathbf{E}\times\mathbf{H}.$$

Derive a relation between $\mathbf{J}\cdot\mathbf{E}$, u and $\mathbf{S}$.

(b) Take all the fields in part (a) to be harmonic so that

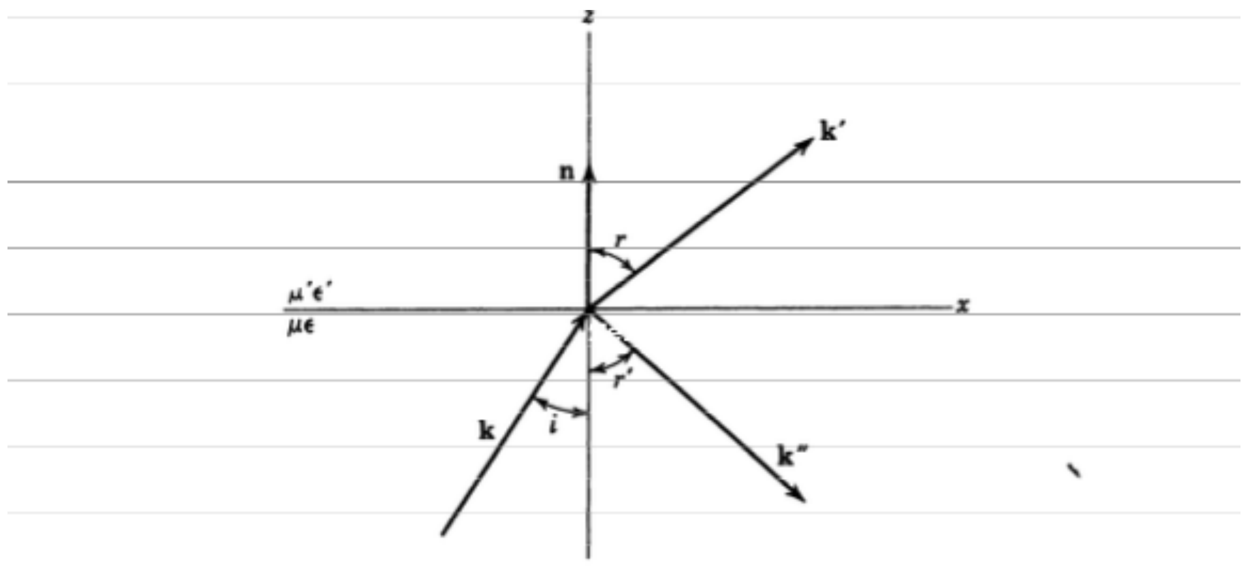
$$\mathbf{E}(\mathbf{x},t)=\tfrac{1}{2}\left(\mathbf{E}(\mathbf{x})e^{-i\omega t}+\mathbf{E}^*(\mathbf{x})e^{i\omega t}\right),\quad \mathbf{D}(\mathbf{x},t)=\tfrac{1}{2}\left(\mathbf{D}(\mathbf{x})e^{-i\omega t}+\mathbf{D}^*(\mathbf{x})e^{i\omega t}\right),$$

$$\mathbf{B}(\mathbf{x},t)=\tfrac{1}{2}\left(\mathbf{B}(\mathbf{x})e^{-i\omega t}+\mathbf{B}^*(\mathbf{x})e^{i\omega t}\right),\quad \mathbf{H}(\mathbf{x},t)=\tfrac{1}{2}\left(\mathbf{H}(\mathbf{x})e^{-i\omega t}+\mathbf{H}^*(\mathbf{x})e^{i\omega t}\right).$$

In terms of

$$w_e=\tfrac{1}{4}\mathbf{E}\cdot\mathbf{D}^*,\quad w_m=\tfrac{1}{4}\mathbf{B}\cdot\mathbf{H}^*$$

and the Poynting vector determine the value of $\mathbf{J}\cdot\mathbf{E}$ as averaged over a single oscillation.



Incident wave **k** strikes plane interface between different media, giving rise to a reflected wave **k**″ and a refracted wave **k**′.

4. Consider two infinite slabs of material with a plane surface at $z = 0$ separating them. The first medium has constant permeability μ and permittivity ϵ so $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. The second medium has constant permeability μ' and permittivity ϵ' with $\mathbf{D}' = \epsilon'\mathbf{E}'$ and $\mathbf{B}' = \mu'\mathbf{H}'$. An incident electromagnetic plane wave with electric field $\mathbf{E}$ and wave vector $\mathbf{k}$ travels in the first medium and strikes the plane interface producing a refracted plane wave in the second medium with electric field $\mathbf{E}'$ and momentum $\mathbf{k}'$ and a reflected plane wave back into the first medium with electric field $\mathbf{E}''$ and momentum $\mathbf{k}''$ with $\mathbf{D}'' = \epsilon\mathbf{E}''$ and $\mathbf{B}'' = \mu\mathbf{H}''$. The three electric fields are

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}, \quad \mathbf{E}' = \mathbf{E}'_0 e^{i\mathbf{k}'\cdot\mathbf{x}-i\omega t}, \quad \mathbf{E}'' = \mathbf{E}''_0 e^{i\mathbf{k}''\cdot\mathbf{x}-i\omega t}$$

All these waves are solutions to the Maxwell equations with no sources. And all of the momentum vectors lie in a plane of incidence that is taken to be the (x, z) plane.

(a) In terms of the electric fields what are the incident, refracted and reflected magnetic fields $\mathbf{B}$, $\mathbf{B}'$ and $\mathbf{B}''$.

(b) In terms of the permeabilities and permittivities how are each of $k = |\mathbf{k}|$, $k' = |\mathbf{k}'|$ and $k'' = |\mathbf{k}''|$ related to the common ω.

(c) $\mathbf{n}$ is in the direction of the normal to the $z = 0$ surface and points into the second medium. The momentum vectors of the three waves respectively make angles i, r and r' with the normal. What are the normal and tangential boundary conditions at the interface between the two media.

(d) Take the incident electric field to be perpendicular to the plane of incidence (i.e., in the y direction). For this incident wave express E'_0 and E''_0 in terms of E_0.

SPHERICAL HARMONICS $Y_{lm}(\theta, \phi)$

$$l = 0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1 \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}} \sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}} \cos\theta \end{cases}$$

$$l = 2 \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}} \sin^2\theta e^{2i\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\dfrac{5}{4\pi}} \left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) \end{cases}$$

$$l = 3 \begin{cases} Y_{33} = -\dfrac{1}{4}\sqrt{\dfrac{35}{4\pi}} \sin^3\theta e^{3i\phi} \\ Y_{32} = \dfrac{1}{4}\sqrt{\dfrac{105}{2\pi}} \sin^2\theta \cos\theta e^{2i\phi} \\ Y_{31} = -\dfrac{1}{4}\sqrt{\dfrac{21}{4\pi}} \sin\theta \left(5\cos^2\theta - 1\right) e^{i\phi} \\ Y_{30} = \sqrt{\dfrac{7}{4\pi}} \left(\tfrac{5}{2}\cos^3\theta - \tfrac{3}{2}\cos\theta\right) \end{cases}$$

Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$

$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \frac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla \cdot \mathbf{A}\, d^3x = \int_S \mathbf{A} \cdot \mathbf{n}\, da \qquad \text{(Divergence theorem)}$$

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

$$\int_V \nabla \times \mathbf{A}\, d^3x = \int_S \mathbf{n} \times \mathbf{A}\, da$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi)\, d^3x = \int_S \phi\mathbf{n} \cdot \nabla\psi\, da \qquad \text{(Green's first identity)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \mathbf{n}\, da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$

$$\int_S \mathbf{n} \times \nabla\psi\, da = \oint_C \psi\, d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial \rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial \phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial \phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial \phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial \rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_2) - \frac{\partial A_1}{\partial \phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial\psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial \phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial \theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial \phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial \theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial\psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial \phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$