Classical Mechanics / Electricity and Magnetism

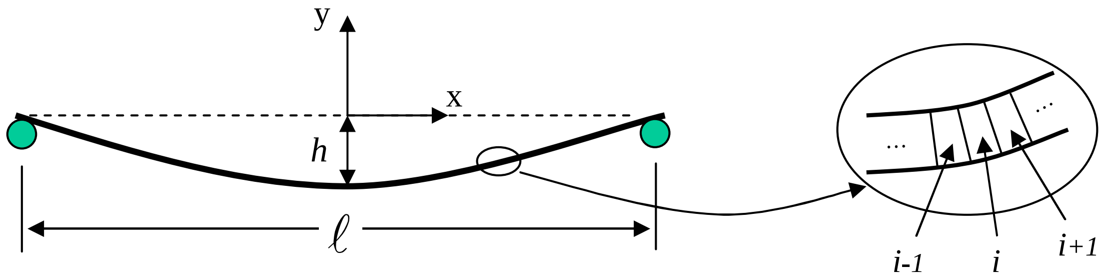
General Exam Questions for August, 2004

Instructions

Answer three questions from each of the two sections, for a total of six problems. Put each of your solutions in a separate answer book. Make sure that you label and sign your name on the cover of each book.

I. Classical Mechanics

1. Consider a strip of paper of width w that lies freely across two parallel horizontal rods a distance ℓ apart as shown in the figure below. The gravitational force mg on the paper causes it to dip slightly in the middle as shown, where $h \ll \ell$ under conditions of static equilibrium. The friction between the rods and the paper can be neglected, so that only its stiffness prevents it from falling.



   (a) The paper is shown chopped up into small segments at the right of the figure. Draw the force diagram for segment i including all forces acting on that segment. To understand the forces between the paper segments you should distinguish between the component of the force that is parallel to the boundary (the shear) and the torques that arise from forces that are normal to the boundary.
   Hint: When the paper bends, the upper surface is being compressed while the lower surface is being stretched. These two forces sum to zero (i.e.. zero tension) but they produce torques which act in an opposite sense on the two boundaries of the segment.

   (b) Write down the conditions for static equilibrium that relate the forces and torques found in part (a).

   (c) Convert the result from (b) into a differential equation for the torque $\tau(x)$ and shear $S(x)$ assuming that the paper has uniform density. Solve for the shape $y(x)$ under the relation

$$\tau(x) = -a\frac{d^2y}{dx^2},$$

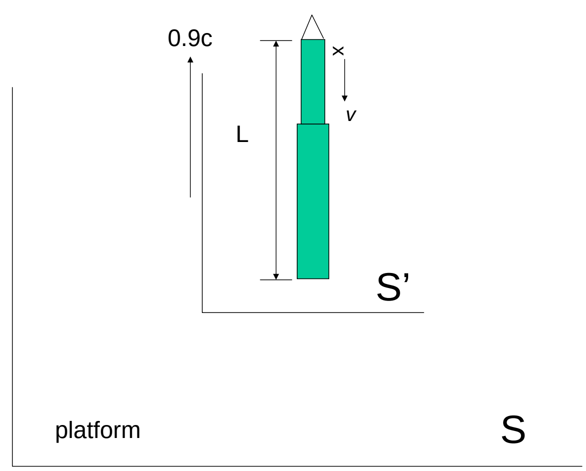
   where a is the stiffness constant of the paper.

2. Consider two coupled degenerate small-oscillation modes in any classical system. With a suitable choice of units, the Hamiltonian can always be cast in the form

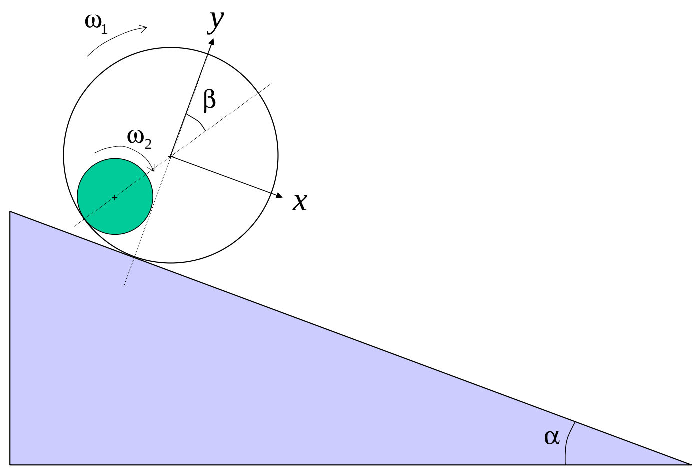
$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}\omega_0^2(q_1^2 + q_2^2) - \kappa^2 q_1 q_2,$$

where q_i and p_i are canonical coordinates and momenta, ω_0 is the common oscillation frequency, and κ is a coefficient proportional to the coupling between the modes. Suppose that at some initial time only mode 1 is excited. Show that in the weak-coupling limit $\kappa \ll \omega_0$ there are times when nearly all of the excitation energy has been transferred to mode 2.

3. A rocket is travelling at a constant velocity v_0 relative to an inertial frame S. The engines are off and an astronaut must do a space walk to repair them. After leaving the cockpit, she must travel from the tip of the rocket to the tail, which is of proper length L. If her speed relative to the rocket is v, how much time elapses in S during the walk? Do not make any approximations.



4. A uniform solid cylinder of radius r_1 and mass m rolls without slipping inside a hollow cylinder of radius $r_2 > r_1$ and the same mass m, as shown in the figure below. The outer cylinder rolls without slipping on a constant slope and accelerates uniformly in the x direction such that the angle β is constant.



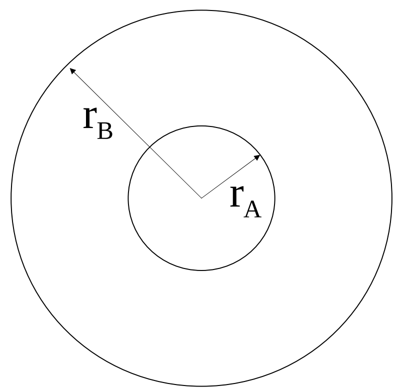
(a) Write down the Lagrangian for this system in terms of coordinates x and β.

(b) Find the acceleration $\ddot{x}$ of the system, assuming that β is a constant of the motion.

(c) Find the value of β such that $\dot{\beta} = 0$ is a solution to the equations of motion. You may leave the answer in the form of a transcendental equation relating β and the constants r_1, r_2, m, α and g.

II. Electricity and Magnetism

1. An inductor consists of two coaxial cylinders of radii r_A and r_B whose helical windings each have n turns per unit length. The length ℓ of both cylinders is large compared to r_B.



   (a) If the two windings are connected independently, what is their mutual inductance M?

   (b) If winding A is connected to an AC voltage source of amplitude V_A, what is the open-circuit voltage V_B across winding B?

   (c) If winding A is connected to an AC current source of amplitude I_A, what is the closed-circuit current I_B in winding B?

2. Consider a very long cylinder of radius a covered with a uniform surface charge density σ and rotating at the angular velocity ω around its axis. For constant ω the fields can be analyzed as a problem in *magnetostatics*.

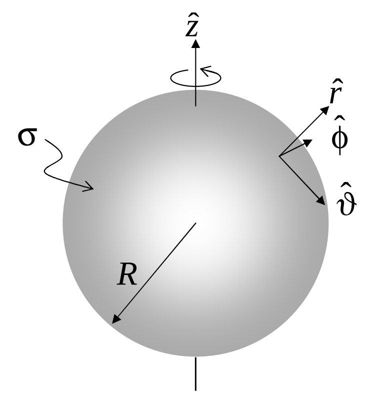
   (a) Show that the current density in this problem is $\vec{J}(\vec{x}) = a\sigma\omega\delta(\rho - a)\hat{e}_\phi$ expressed in the usual cylindrical coordinates ρ, ϕ, z.

   (b) By symmetry the magnetic field must be of the form $\vec{H}(\vec{x}) = H(\rho)\hat{e}_z$ everywhere. In regions where the charge density is zero, show that $\vec{H}$ may be expressed as the gradient of a scalar potential $\phi(\vec{x}) = z\Phi(\rho, \phi)$, where $\Phi(\rho, \phi)$ satisfies the two-dimensional Laplace equation. The general set of solutions to the two-dimensional Laplace equation are: 1; $\ln\rho$; $\rho^m \cos m\phi$, $\rho^m \sin m\phi$, $\rho^{-m} \cos m\phi$, $\rho^{-m} \sin m\phi$, $m = 1, 2, \ldots$.

   (c) On the basis of part (b), find the magnetic field inside the cylinder.

3. An insulating spherical shell of radius R carries a charge Q, which is uniformly distributed over its surface. The sphere rotates about the z axis with angular velocity ω. In regions where the current is zero, the magnetic field is given by the gradient of the magnetic potential $\vec{B} = \vec{\nabla}\Phi_m$, where

$$\Phi_m(\vec{x}) = \begin{cases} \frac{2}{3}\frac{\mu_0}{4\pi}Q\omega\frac{r}{R}\cos\theta & : \quad r < R \\ -\frac{1}{3}\frac{\mu_0}{4\pi}Q\omega\frac{R^2}{r^2}\cos\theta & : \quad r > R \end{cases} .$$



(a) What are the electric and magnetic fields inside and outside the sphere? Express the answer in the spherical basis $(\hat{r},\, \hat{\theta},\, \hat{\phi})$ noting that

$$\vec{\nabla} = \hat{r}\left(\frac{\partial}{\partial r}\right) + \frac{\hat{\theta}}{r}\left(\frac{\partial}{\partial\theta}\right) + \frac{\hat{\phi}}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\right)$$

(b) Assuming that ω is increased from 0 in a quasi-static way, use Faraday's law to calculate the induced electric field at $r = R$ as a function of θ.

(c) Calculate the angular momentum $\vec{L}$ stored in the fields, and the torque $\vec{\tau}$ which the electric field produces on the sphere. Verify that classical expressions are obeyed, that is

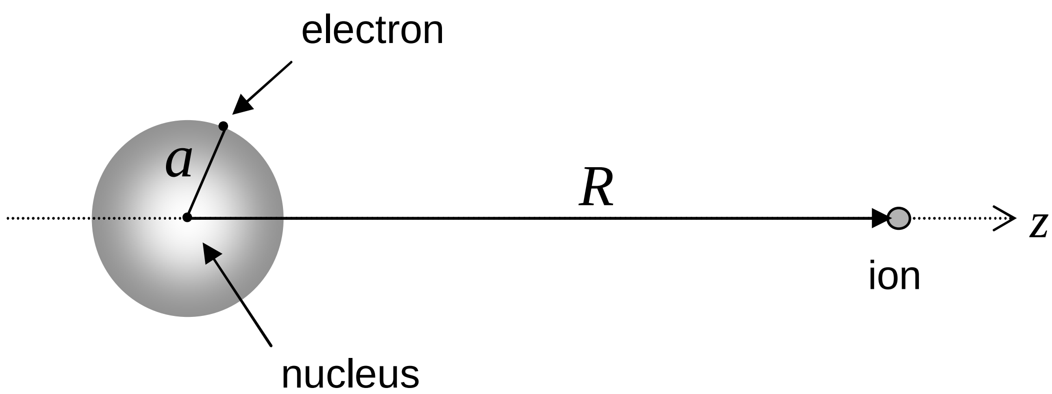
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

and $\vec{L} = I\omega\hat{e_z}$ where I is the moment of inertia of the electromagnetic fields. Express I in terms of Q, R, etc.

Hint:

$$\begin{aligned} \vec{L} &= \epsilon_0 \int \vec{x} \times (\vec{E} \times \vec{B})\, d^3x \\ \vec{\tau} &= \int \vec{r} \times d\vec{F} \end{aligned}$$

4. A neutral hydrogen atom is represented classically by a conducting sphere of radius a surrounding the nucleus as shown in the figure below. The atom is polarized in the presence of an ion of charge q located a distance $R \gg a$ from the atom.



(a) Derive the multipole expansion of the electrostatic energy W_{ext} of a charge distribution $\rho(\vec{x})$ in an external electric field, using

$$W_{\text{ext}} = \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x$$

where $\Phi(\vec{x})$ is the potential corresponding to the external electric field. You should carry out the expansion to the quadrupole term.

(b) Solve for the surface charge density on the sphere $\sigma(\theta)$ in the presence of the ion.

(c) Use the above results to estimate the interaction energy between the ion and the atom.

(d) Calculate the polarizability α of the atom using the following equation for the definition of α,

$$W_{\text{ext}} = -\,\alpha \frac{q^2}{R^4}.$$