CLASSICAL MECHANICS/ ELECTRICITY AND MAGNETISM

Preliminary Examination

January 16, 2008

9:00 - 15:00 in P-121

Answer a total of **SIX** questions, choosing **THREE** from each section. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book. Make sure you clearly indicate who you are, and the problem you are answering. Double-check that you include everything you want graded, and nothing else.

SECTION A - Classical Mechanics

A1. A broomstick 3 m long is to pass through a garage 3 m long, as shown in Figure 1. The garage has a special door arrangement. Just when the leading end of the broomstick coincides with the entry door, it quickly opens. Then when the leading end of broomstick coincides with the other door, the door involved opens and the other door closes. The diagrams below show the observations made by observers at rest in the inertial system of the garage and by observers at rest in the broomstick's inertial system. How much of the broomstick will be cut when the entrance door closes if $v = 0.8c$? Justify your answer by considering Lorentz transformation between the two inertial systems.
(Hint: consider the coordinates of the two events in both frames: A. The exit door opens, B. The entrance door closes.)

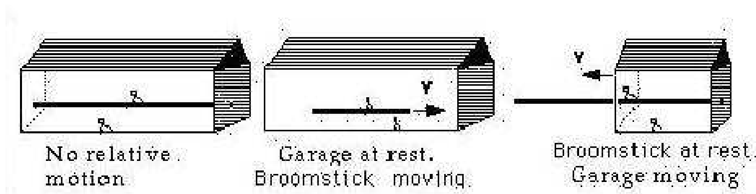

Figure 1: For problem **A1**.

A2. Consider the following equations of motion of a particle on a plane

$$\ddot{x} = B\dot{y}; \quad \ddot{y} = -B\dot{x}$$

where B is a constant.

(a) Is this system conservative? *i.e.,* can these equations be obtained from a Lagrangian by using the minimal action principle?

(b) Find a conserved Hamiltonian which leads to these equations of motion. Is it unique?

(c) The system is translationally invariant. This means that the two dimensional momentum is conserved. In other words, there are two functions of coordinates (x, y) and canonical momenta (p_x, p_y) which have vanishing Poisson brackets with the Hamiltonian. Find the two conserved components of momentum and show using the Poisson brackets that they indeed generate translations.

A3. The bearing of a rigid pendulum of mass m is forced to rotate uniformly with angular velocity ω as shown in Figure 2. The angle between the axis of rotation and the pendulum is θ. Neglect the inertia of the bearing and of the rod connecting it to the mass. Neglect friction. Include the effects of the uniform gravity.

(a) Find the differential equation for θ .

(b) At what rotation rate ω_c does the stationary point at $\theta = 0$ become unstable?

(c) For $\omega > \omega_c$, what is the stable equilibrium value of θ?

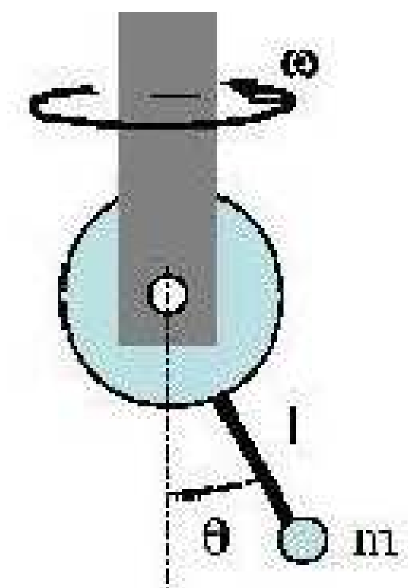

Figure 2: For problem **A3**.

A4. Consider the driven damped harmonic oscillator described by the equation of motion

$$\ddot{x} + 2\,\dot{x} + \left(1 + \kappa^2\right) x = f(t)$$

subject to the initial conditions $x(0) = 0 = \dot{x}(0)$.

(a) Use the solutions of the associated homogeneous problem to construct the corresponding Green's function $G(t;t')$, and verify that it can be written as a function of the time difference $(t - t')$ only.

(b) Find the explicit solution when the driving force $f(t)$ is the unit step function.

(c) Confirm your solution by taking the Laplace transform of it, and comparing with the Laplace transform obtained directly from the original equation.

SECTION B - Electricity and Magnetism

B1. Consider an electric charge density of the following form:

$$\rho = \begin{cases} Ar\cos\theta & , \quad \text{for } 0 \le r < a \\ 0 & , \quad \text{for } r \ge a. \end{cases}$$

Find the electrostatic potential <u>inside and outside</u> the charge distribution, given that both the potential and its radial derivative are continuous everywhere.

B2. Electrostatic charge is distributed in a sphere of radius R centered on the origin. Determine the form of the resultant potential $\phi(\vec{r})$ at distances much greater than R, as follows:

(a) Express the solution of

$$\vec{\nabla}^2\phi = -\frac{\rho(\vec{r})}{\epsilon_0}$$

in the form of an integral over all space.

(b) Show that for $r \gg r'$,

$$|\vec{r} - \vec{r}'| = r - \frac{\vec{r}\cdot\vec{r}'}{r} + O\left(\frac{1}{r}\right).$$

(c) Use your results in (a) and (b) to show that $\phi(\vec{r})$ has the long-distance form

$$\phi(\vec{r}) = \frac{M}{r} + \frac{\vec{d}\cdot\vec{r}}{r^3} + O\left(\frac{1}{r^3}\right).$$

Identify M and $\vec{d}$.

B3. (a) Write down Maxwell's equations for the propagation of an electromagnetic wave of frequency ω along a long straight waveguide which has perfectly conducting walls and whose interior is a dielectric of permeability μ and permittivity ϵ. Let the direction along the waveguide be the z direction.

(b) Consider the propagation of a TEM mode (*i.e.*, one for which $E_z = B_z = 0$). Explain why the propagation of such a mode is impossible if the waveguide is hollow, but is possible for a coaxial waveguide.

B4. X-rays which strike a metal surface at an angle of incidence to the normal greater than a critical angle θ_0 are totally reflected. Assuming that a metal contains n free electrons per unit volume, calculate θ_0 as a function of the frequency ω of the X-rays. The metal occupies the region $x > 0$ (shaded region in the picture). The X-rays are propagating in the x-y plane, as shown in Figure 3, and their polarization vector is in the z-direction out of the page.

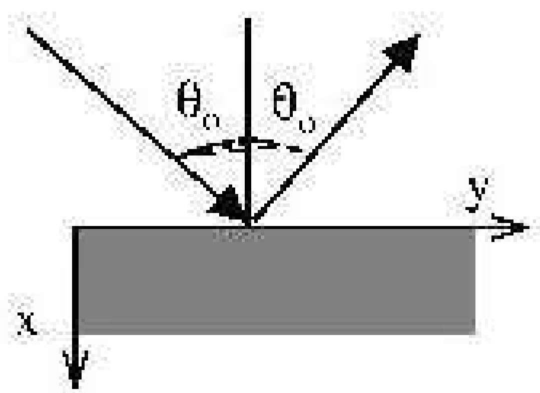

Figure 3: For problem **B4**.