Classical Mechanics/Electricity and Magnetism
General Exam
January 11, 2006
09:00 - 15:00 in P-121

Answer three (3) questions from each of the two (2) sections for a total of six (6) solutions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented separately in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are, and what is the problem you are answering. Double-check that you include everything you want graded, and nothing else.

Section 1 — mostly mechanics

1. Two athletes run on a track in the z direction. The lanes they run in are separated by a distance Δx in the x direction. The first runner crosses the finish line by time ΔT earlier than the second one.

(a) Is there an inertial system K' in which the two runners cross the finish line simultaneously? Does your answer depend on the values of Δx and ΔT?

(b) If such a system K' exists, find explicitly the Lorentz transformation between the original system K and K'. If in the system K the two athletes started running simultaneously, who started off earlier and by how much from the point of view of an observer moving with K'?

2. **Kinematic Doppler width reduction.**

(a) The Doppler broadening of an optical transition is caused by the inhomogeneous distribution of Doppler shifts for atoms moving at different speeds; to an atom moving in the propagation direction of light at the speed v, the frequency of light appears downshifted by the fraction v/c. Assume that a sample of atoms with the rest-frame resonant frequency ω_0 is illuminated by light propagating along $\hat{\mathbf{z}}$. For simplicity, assume that the atoms have a uniform distribution of momenta along the z axis over the interval $[-p, p]$, with kinetic energies ranging up to E. Find the corresponding Doppler width $\Delta\omega$.

(b) Now a fast atomic beam is formed by passing the entire sample of atoms through an accelerating potential, also directed along $\hat{\mathbf{z}}$, that adds the energy E_0 to each atom. Assume that $E_0 \gg E$, but that the motion is still non-relativistic. In this approximation, solve for the corresponding spread in velocities Δv_z and for the resulting Doppler width $\Delta\omega$. By what ratio are these quantities reduced compared with the unaccelerated sample in part (a)?

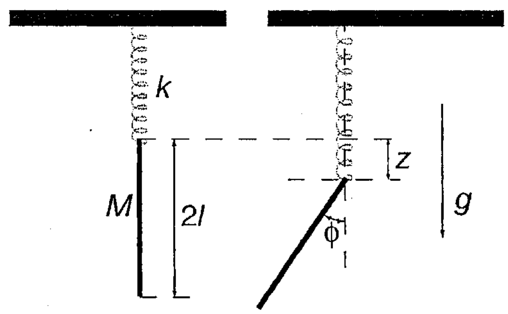

Figure 1: Drawing for problem 3.

3. A uniform bar of mass M and length $2l$ is suspended from one end by a spring of force constant k; see the figure. The bar can swing freely only in one vertical plane, and the spring is constrained to move only in the vertical direction along gravity. Find the small-oscillation frequencies around the stationary configuration.

4. A small charged object (mass m, charge q) is moving in a magnetic field of the form $\mathbf{B}(t) = B(t)\,\hat{\mathbf{e}}_z$. If the variation of the magnetic field is so slow that light has ample time to propagate across the experimental setup during any characteristic time such as the inverse of the cyclotron frequency, all electromagnetic fields on the particle may be derived from a vector potential $\mathbf{A}(\mathbf{r},t) = \frac{1}{2}[\mathbf{B}(t) \times \mathbf{r}]$. This is known as the quasistatic approximation. The motion along the direction of the magnetic field separates, and need not be considered any further.

(a) The Hamiltonian for the system in terms of the Cartesian coordinates x and y (collectively, $\mathbf{x}$) and their corresponding canonical momenta ($\mathbf{p}$) reads

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 .$$

Please explain.

(b) Find the expression of the kinetic energy $m\dot{\mathbf{x}}^2/2$ in terms of canonical coordinates and momenta.

(c) Is kinetic energy a constant of the motion? What's the physical reason?

Section 2 — mostly E&M

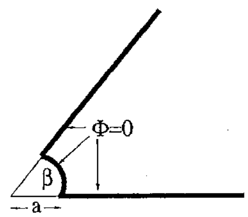

Figure 2: Drawing for problem 5.

5. The two dimensional region $a \leq \rho < \infty$, $0 \leq \phi \leq \beta$ is bounded by conducting surfaces at $\phi = 0$, $\phi = \beta$ and $\rho = a$ held at zero potential, $\Phi = 0$ as indicated in the figure. At large distances there are charges, so that the potential is not identically zero.

(a) Write down the general solution for the potential $\Phi(\rho, \phi)$ in the neighborhood of $\rho = a$, taking into account the appropriate boundary conditions.

(b) Find the form (direction and dependence on the angle ϕ) of the electric field on the surface $\rho = a$. Assume that a is the smallest relevant length scale in the problem, and that the configuration of the far-away charges has no particular symmetry.

Recall that, as long as one does not require that the solution has the period 2π in the angle ϕ, any constant and all of the functions $\rho^{\pm\nu} \sin \nu\phi$, $\rho^{\pm\nu} \cos \nu\phi$, $\ln \rho$, ϕ, and $\phi \ln \rho$ are solutions to the 2D Laplace equation for an arbitrary (even complex) number ν.

6. **Free electron plasma frequency and pondermotive energy.**

(a) A free electron is placed in an electromagnetic plane wave. Provided the wavelength is long enough for our purposes, the electric field is given by

$$\mathbf{E} = E_0 \hat{\mathbf{e}}_x \cos \omega t = \hat{\mathbf{e}}_x \frac{E_0}{2} (e^{-i\omega t} + \text{c.c.}) ,$$

where $\hat{\mathbf{e}}_x$ denotes the direction of polarization. Assuming the motion is nonrelativistic, find the position $\mathrm{x}(t)$ of the oscillating electron, and the average "quiver energy" $E_p = \frac{1}{2} m \langle \mathrm{v}^2 \rangle$. In high-field physics this energy is called the *pondermotive energy* or frequently, but misleadingly, the *pondermotive potential.* Assume here that the electron started at rest and the electric field was turned on slowly, so that there is no drift velocity and the time averaged position did not change when the field was turned on.

(b) Now consider the electron to be part of a free-electron or "Drude model" of a metal or plasma, with a density of n electrons per unit volume. Assuming that the electrons are stationary without the external field, find the macroscopic polarization $\mathrm{P}(t)$ associated with the oscillating dipole moments $\mathbf{d}(t)$ of the electrons. Use this to find the dielectric constant of the medium, or equivalently the index of refraction. At what frequency ω_p does the index of refraction equal zero? (Hints: This is one way to derive the plasma frequency of the free electron gas. As the electrons are not bound, you do not have to worry about local-field corrections. Nonetheless, the lattice of ions in a metal provides a neutralizing background charge, so that it makes sense to talk about the dipole moment associated with a moving electron.)

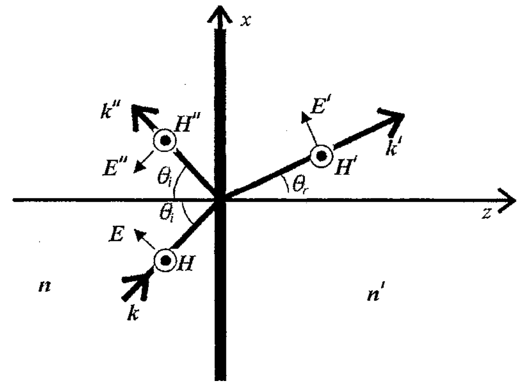

Figure 3: Drawing for problem 7.

7. **Evanescent waves and gradient forces.** A plane wave is incident on a dielectric interface with its electric field in the plane of incidence, as shown in the figure. You are given that the angles of incidence and refraction satisfy Snell's law $\sin\theta_i/\sin\theta_r = n'/n$, and for a generic plane wave of the form $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ the ratio of transmitted and incident field amplitudes is

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta_i}{n'\cos\theta_i + n\cos\theta_r}.$$

(a) Now consider the case of total internal reflection with an angle of incidence just above the critical angle, so that the angle of refraction obtained from Snell's law is imaginary. The electric field is nevertheless non-zero in a small region beyond the dielectric interface. We can define two real parameters to characterize the system, $A \equiv \sin\theta_r > 1$ (but $A \simeq 1$), and $B \equiv i\cos\theta_r$ (and $0 < B \ll 1$). Solve for the electric field on the far (n') side of the interface as a function of these parameters, by writing out the wave vector $\mathbf{k}'$ explicitly as a function of the parameters and the incoming frequency. Show that the wave propagates mostly along $\hat{\mathbf{x}}$ and that it is exponentially damped as a function of the distance z from the surface; hence the term *evanescent* wave. Find the amplitude of the wave, assuming (as above) that A is close to unity and B is small.

(b) Even though the evanescent wave transmits no power, it can exert forces. In fact, if the light is tuned "far" (several natural linewidths) to the short-wavelength side of an atomic resonance, the atom experiences a *gradient force* that is proportional to the gradient of the energy density of the electromagnetic field, and points toward lower energy densities. What, then, happens to a slow enough atom when it hits the evanescent wave?

8. **Uniformly magnetized bodies.**

(a) Consider a situation when there are no free currents and a constant (in space and time) magnetization $\mathbf{M}$ inside an arbitrary shape object. Since the magnetization cuts off abruptly at the surface, there will be a singularity in the magnetization current $\nabla \times \mathbf{M}$. Show that $\nabla \times \mathbf{M} = \mathbf{M} \times \hat{\mathbf{n}}\,\delta(z)$, where $\hat{\mathbf{n}}$ is the outward normal and z is a local Cartesian coordinate along $\hat{\mathbf{n}}$ such that $z = 0$ is on the surface and $z > 0$ is on the outside.

(b) As a result, the magnetic induction field $\mathbf{B}$ is exactly the same as if there were no magnetization, but instead a surface current density $\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}}$. Supply a simple argument.

(c) Demonstrate that a rotating sphere with a constant surface charge density and nonmagnetic material inside gives off the same magnetic induction field as a stationary sphere with no free charges and a certain magnetization inside.