Classical Mechanics / Electricity and Magnetism

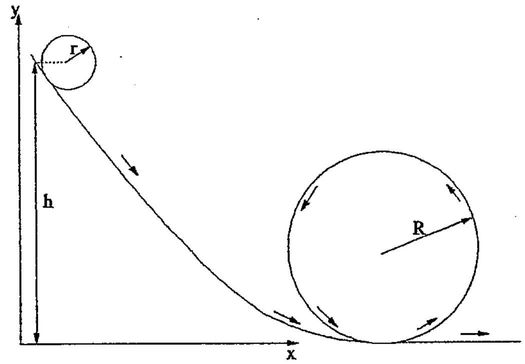
General Exam Questions for January, 2004

Instructions

Answer three questions from each of the two sections, for a total of six problems. Put each of your solutions in a separate answer book.

I. Classical Mechanics

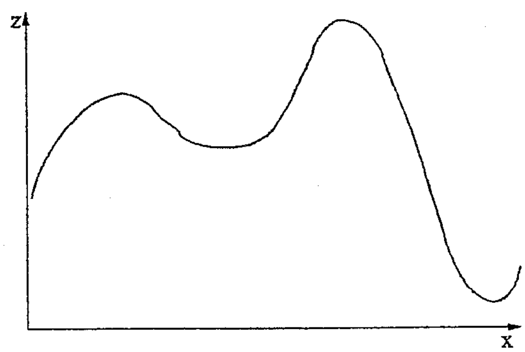
1. Consider a uniform solid sphere that rolls without slipping on a curved track in the vertical plane as shown in the figure at the right. In order to emerge on the horizontal section of track at the right the sphere must roll down the slope and around the inside of the loop of radius R without leaving the track. The sphere of radius $r < R$ and mass m is released from rest at height h as shown.



   (a) Find the minimum height h required to prevent the sphere from falling off the track in the loop.

   (b) Discuss the physical meaning of the result for h in the case where $0 < (R-r) \ll R$.

2. A bead of mass $m = 1$ is constrained to move in a vertical plane along a wire shaped according to a curve $z = z(x)$ as shown in the figure. The distance s traveled by the bead along the wire is given by the relation



$$ds^2 = dx^2 + dz^2$$

so that there are mappings $x \leftrightarrow s \rightarrow z$. The bead moves without friction under the influence of gravity which acts in the $-\hat{z}$ direction with a uniform acceleration of 1.

   (a) Write the Lagrangian for the system, using x as the generalized coordinate.

   (b) Show that the Lagrangian is $L = \frac{1}{2}\dot{s}^2 - z(s)$ when the distance s is used as the generalized coordinate.

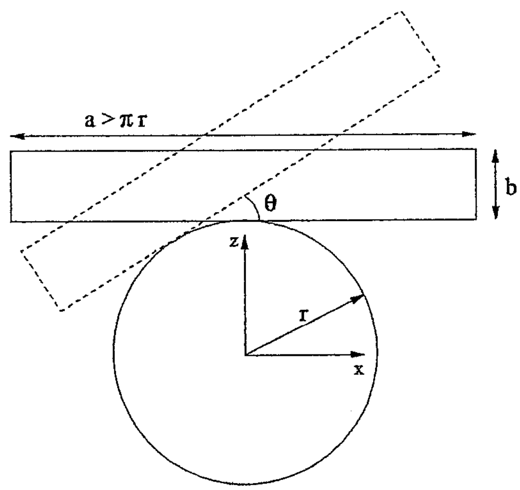
   (c) Suppose that the curve $z(x)$ is arranged such that $z(s) = \frac{1}{2}s^2$ in some interval of s including $s = 0$. If the body is released from rest, show that it will take the same time to fall to the point $s = 0$ from any point $s \neq 0$ within this interval.

   (d) Derive the following differential equation that must be satisfied by any function that meets the condition $z(s) = \frac{1}{2}s^2$ within some interval,

$$\frac{dz}{dx} = \pm\left[\frac{1}{2z} - 1\right]^{-\frac{1}{2}}.$$

   Warning: Do not try to find an explicit form for $z(x)$.

3. A uniform rectangular block of mass m and dimensions $a \times b \times c$ rests on a horizontal cylinder of radius r as shown in the figure, such that the cylinder contacts the block at its midpoint when the block is horizontal ($\theta = 0$). The cylinder is held fixed but the block is free to rock back and forth without slipping on the cylinder.



(a) Show that the coordinates of the center of gravity of the block are given by the following relations in a coordinate system with the origin at the center of the cylinder.

$$x = r(\theta \cos\theta - \sin\theta) - \frac{b}{2}\sin\theta$$

$$z = r(\theta \sin\theta + \cos\theta) + \frac{b}{2}\cos\theta$$

(b) What range of values are allowed for r, a and b such that $\theta = 0$ is a stable equilibrium, assuming a uniform gravitational acceleration g pointing down in the figure.

(c) Under the conditions specified in part (b), what is the frequency of small oscillations?

4. Consider a classical body of charge q and mass m moving under the influence of electromagnetic fields specified by the scalar and vector potentials $\phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$.

(a) What is the Hamiltonian for this system, treating the electromagnetic field as external and fixed?

(b) Show that Hamilton's equations of motion lead to an expression for the canonical momentum $\vec{p}$ that is different from $m\vec{v}$.

(c) Write down the Lagrangian for this system and show that the principle of least action leads to the Lorentz force.

The following vector relation may be useful.

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

II. Electricity and Magnetism

1. Consider a classical model of an electron as a negative charge $-e$ uniformly distributed within a sphere of radius R.

   (a) Calculate the electric field $\vec{E}$ everywhere inside and outside the sphere. *Hint:* Use Gauss' law (SI units)
   $$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}.$$

   (b) Find the electrostatic potential energy W of this charged sphere.

   (c) By setting W equal to the rest energy of the electron mc^2, find the value of R. Express your answer in units of the classical electron radius r_0 defined as
   $$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}.$$

2. Consider a very long cylinder of radius a made from an insulating material with permeability μ. The cylinder is placed in a constant magnetic field that was uniform through all space before the cylinder was introduced. We are interested in the steady state after the cylinder is placed with its axis perpendicular to the direction of the field, and any transients have disappeared.

   (a) Show that $\vec{\nabla} \cdot \vec{H} = 0$ holds everywhere except on the surface of the cylinder.

   (b) Find the magnetic field inside the cylinder.
   The general solution to the Laplace equation in two dimensions can be expanded in terms of the functions $f_m(\rho, \phi)$ and $g_m(\rho, \phi)$ written in polar coordinates as
   $$f_m(\rho,\phi) = \rho^m \cos(m\phi)$$
   $$g_m(\rho,\phi) = \begin{cases} \ln \rho & \text{if } m = 0 \\ \rho^m \sin(m\phi) & \text{otherwise} \end{cases}$$
   for all integers m, both positive and negative.

3. A particle of charge q and mass m moves in the xy plane under the influence of a magnetic field $\vec{B} = B\hat{z}$.

   (a) Write down the force on the particle and solve for an orbit assuming the initial momentum is $p_0\hat{x}$.

   (b) Add a small transverse electric field $\vec{E} = \varepsilon\hat{x}$ and solve for the resulting perturbation that it causes to the orbit found in part (a).
   Hint: Transform the fields to a frame moving at a constant velocity in the plane transverse to the magnetic field, and look for a choice of the velocity that makes the electric field disappear. Solve for the orbit in this frame, then transform it back to the stationary frame.

4. Consider a collection of charged particles with charges $q^{(n)}$ and positions $\vec{\mathrm{R}}+\vec{\mathrm{r}}^{(n)}(t)$. The total charge of the collection is zero $\sum_n q^{(n)} = 0$, but in general it has a nonzero dipole moment $\vec{\mathrm{d}} = \sum_n q^{(n)}\vec{\mathrm{r}}^{(n)}(t)$. To make the dipole approximation we take the displacements $\vec{\mathrm{r}}^{(n)}$ to be very small. The charged particles are subject to an electric field $\vec{\mathrm{E}}(\vec{\mathrm{r}},t)$ that varies in space and time. In this problem all time-dependent quantities oscillate with the same characteristic frequency ω. Latin letters stand for the time-dependent quantities, while their script or Greek counterparts are the corresponding time-independent complex amplitudes, as follows;

$$\begin{aligned} \vec{\mathrm{r}}(t) &= \frac{1}{2}[\vec{\rho}e^{-i\omega t} + \vec{\rho}^{*}e^{i\omega t}], \\ \vec{\mathrm{d}}(t) &= \frac{1}{2}[\vec{\delta}e^{-i\omega t} + \vec{\delta}^{*}e^{i\omega t}], \\ \vec{\mathrm{E}}(\vec{r},t) &= \frac{1}{2}[\vec{\mathcal{E}}(\vec{r})e^{-i\omega t} + \vec{\mathcal{E}}^{*}(\vec{r})e^{i\omega t}]. \end{aligned}$$

(a) The electric field in the neighborhood of the charged particles is approximated as

$$\vec{\mathcal{E}}(\vec{\mathrm{R}}+\vec{\mathrm{r}}^{(n)}) \simeq \vec{\mathcal{E}}(\vec{\mathrm{R}}) + \mathrm{r}_j^{(n)}\frac{\partial}{\partial \mathrm{R}_j}\vec{\mathcal{E}}(\vec{\mathrm{R}}),$$

with the sum over j implied. Show that the net force on the system of charges, when averaged over one cycle (over time $2\pi/\omega$), is

$$\vec{\mathrm{F}}_E = \frac{1}{2}\Re\left\{(\vec{\delta}^{*}\cdot\vec{\nabla})\vec{\mathcal{E}}(\vec{\mathrm{R}})\right\}.$$

(b) Because the particles are moving, they are also subject to the time-dependent magnetic field $\vec{\mathrm{B}}(\vec{\mathrm{r}},t)$. Show that the cycle-averaged net magnetic force on the system is given by

$$\vec{\mathrm{F}}_B = \frac{1}{2}\Re\left\{i\omega\vec{\delta}^{*}\times\vec{\mathcal{B}}(\vec{\mathrm{R}})\right\}.$$

(c) Show that the amplitudes of the electric and magnetic fields satisfy the relation

$$\vec{\mathcal{B}}(\vec{\mathrm{r}}) = \frac{\vec{\nabla}\times\vec{\mathcal{E}}(\vec{\mathrm{r}})}{i\omega}.$$

(d) Show that the total cycle-averaged force on the dipole is given by

$$\mathrm{F}_i(\vec{\mathrm{R}}) = \frac{1}{2}\Re\left\{\delta_j^{*}\frac{\partial}{\partial \mathrm{R}_i}\mathcal{E}_j(\vec{\mathrm{R}})\right\}.$$

This is the force that makes laser cooling and trapping work.