Preliminary Exam: Classical Mechanics, Monday August 24, 2020. 9:00-12:00

Answer a total of any **THREE** out of the four questions. If a student submits solutions to more than three problems, only the first three problems as listed on the exam will be graded. Students should write their solutions on blank 8.5 by 11 paper or in a blue book, putting their name on each page, the number of the problem and the number of the page in their solution on each page (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their solutions in sequence using a cell phone or a scanner and email them in a file or files to the prelim committee chair philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam. (It might be easier to transfer the files to a laptop first.) Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems. The chair will immediately check if the emailing is readable or if a resend is required.

1. An object of unit mass orbits in a central potential $U(r)$. The equation of the orbit is given by

$$r = ce^{-b\theta}$$

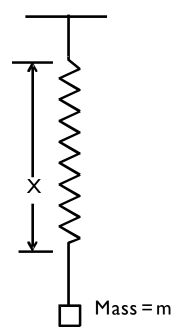
where θ is the azimuthal angle measured in the plane of orbit, and b and c are constants.

(**a**) With $\vec{r}$ measured from the origin of coordinates, show that the angular momentum $\vec{L}$ is conserved.

(**b**) To within a multiplicative constant find the $U(r)$ that would produce this $r = ce^{-b\theta}$ orbit assuming that $U(r)$ goes to 0 as r goes to infinity.

(**c**) In terms of the multiplicative constant in part **(b)**, for what value L_0 of $L = |\vec{L}|$ would the orbit be circular?

(**d**) What is the shape of the orbit if L is (i) less than, and, (ii) greater than the critical value L_0?



2. As shown in the figure, a mass m is suspended in equilibrium at time $t < 0$ in the earth's gravitational field by a vertical massless spring that exerts a force $F = -k(x - \ell)$, where x is the length of the spring and ℓ is its length when $F = 0$. The acceleration due to gravity can be taken to be a constant of magnitude g. At time $t = 0$ the point of contact, to which the upper end of the spring is attached, begins to oscillate sinusoidally up and down with amplitude A and frequency ω. The driving frequency ω is off resonance, i.e. ω is not equal to $\omega_0 = (k/m)^{1/2}$.

(**a**) What is the equilibrium value x_0 of x at $t < 0$?

(**b**) Set up the equation of motion for $x(t)$ for $t > 0$.

(**c**) Solve the equation of motion for $x(t)$ for $t > 0$.

3. The Lagrange function of a one-dimensional classical non-relativistic system is given by

$$L_1(q, \dot{q}, t) = \frac{1}{2}\, m\, \dot{q}^2\, e^{\alpha t} - \frac{1}{2} k\, q^2\, e^{\alpha t}$$

where α and k are positive constants.

(a) Derive the equations of motion of the system. Which system do these equations of motion correspond to?

(b) Derive from the Lagrange function $L_1(q, \dot{q}, t)$ a new Lagrange-function $L_2(Q, \dot{Q}, t)$ where the new coordinate Q is related to q by $Q = q\, e^{\gamma t}$. Choose the constant γ such that $L_2(Q, \dot{Q}, t) = L_2(Q, \dot{Q})$, i.e. the explicit time-dependence is removed in L_2.

(c) Derive the equations of motion of the time-independent Lagrangian L_2 derived in part **(b)**, and solve them for (i) $k > m\alpha^2/4$, (ii) $k = m\alpha^2/4$, (iii) $k < m\alpha^2/4$ for general initial conditions.

(d) Using the solutions for $Q(t)$ obtained in part **(c)** invert the transformation introduced in part **(b)** to derive the corresponding forms for $q(t)$ in each of the three cases described in part **(c)**.

4. Let a system with $1 \leq k \leq n$ degrees of freedom be described by the Hamiltonian $H = H(p_k, q_k)$.

(a) If the quantity $f = f(p_k, q_k)$ has no explicit time-dependence, i.e. $\partial f/\partial t = 0$, show that $df/dt = [f, H]$, where

$$[u, v] = \sum_{k=1}^{n} \left(\frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right)$$

denotes the $[u, v]$ Poisson bracket.

(b) In a 3-dimensional system the Hamiltonian is given by

$$H = \frac{\vec{p}^2}{2m} + a|\vec{q}|^b$$

where a, b are constants. Use the Poisson bracket condition derived in part **(a)** to determine the values of the constants a and b for which the vector

$$\vec{V} = \vec{p} \times \vec{L} + c\, \frac{\vec{q}}{|\vec{q}|}$$

is conserved, where $\vec{L} = \vec{q} \times \vec{p}$ denotes the angular momentum and c is a constant.