Preliminary Exam: Classical Mechanics, Monday August 19, 2019. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a **SEPARATE** blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. The orbit of a particle moving under the influence of a central force potential $V(r)$ is $r\theta =$ constant. Assume that the motion is non-relativistic and that the potential is zero for infinitely large r.

   (**a**) Using Newton's Laws of motion derive the equations of motion in polar coordinates.

   (**b**) Prove that the angular momentum and the total energy are conserved.

   (**c**) Determine the most general form for the potential $V(r)$ that can produce this orbit subject to its approaching zero at large r.

2. A one-dimensional anharmonic oscillator consists of a mass m in a potential given by

$$U(x) = \frac{1}{2}kx^2 - \frac{1}{3}m\lambda x^3 + Ex\cos(\omega t),$$

   where λ is small, i.e. the second term in the potential is small compared to the first term in the potential. All of k, λ and E are constants. Assume initial conditions $x = a$ at $t = 0$ and $dx/dt = 0$ at $t = 0$.

   (**a**) Solve the equation of motion for $\lambda = 0$ and $E = 0$.

   (**b**) Solve the equation of motion to first order in λ for $E = 0$ and the above initial conditions. Hint: Use a first order solution of the form $x(\lambda) = x(\lambda = 0) + \lambda x_1$ where $x(\lambda = 0)$ is the solution from part (a) and substitute this in the equation of motion and solve for x_1. Note that in the final solution the equilibrium value of x_1 is not zero.

   (**c**) Solve the equation of motion for $\lambda = 0$, when E is not zero.

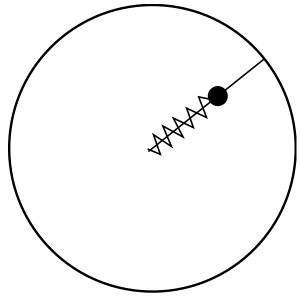

Figure 1: A disk with a mass on a spring: one end of the spring is fixed at the center of the disk. The string has unstretched length ℓ, and expands along one of the radii towards the edge of the disk.

3. A disk of mass M and radius R rotates in a horizontal plane about its center. A point mass m can slide freely along one of the radii of the disk, and is attached to the center of the disk by a massless spring of natural length ℓ and force constant k, as shown in the figure.

   (**a**) When the mass is at a distance r from the center of the disk find the moment of inertia of the system (disk, spring and mass combined) about an axis through the center of the disk perpendicular to the disk.

   (**b**) When the mass is at a distance r from the center of the disk find an expression for the energy of the system in terms of r, $\dot{r}$, and the total angular momentum J of the system.

   (**c**) Derive Lagrange's equations.

   (**d**) Suppose the disk is rotating at a constant angular velocity Ω_0 with the spring having a constant extension $r = r_0$ during this rotation. Find r_0 as a function of Ω_0.

   (**e**) Find the frequency of small oscillations around the equilibrium configuration.

4. A point particle of mass m moves in one dimension subject to the potential

$$V(x) = \frac{a}{\sin^2 \frac{x}{x_0}}$$

   with constant a and x_0.

   (**a**) Solve for Hamilton's characteristic function $W(x, \alpha)$ for this potential, obtaining $W(x, \alpha)$ as an indefinite integral that you do not need to evaluate.
   Hint: W is the solution to the Hamilton-Jacobi equation for fixed total energy α, written as $H(x, \partial W/\partial x) - \alpha = 0$.

   (**b**) In the angle-action variable method, a new constant of the motion J is introduced which measures the change in W, viz. $\oint dW$, over a single period of the motion. Under what conditions is this method applicable? Justify its use for this potential.

   (**c**) Evaluate J for this potential, given total energy α.
   Hint: you may find the following definite integral to be useful.

$$\int_{-\phi_0}^{\phi_0} \sqrt{b^2 - \sec^2 \phi}\, d\phi = \pi(b - 1)$$

   where ϕ_0 are the zeros of the integrand.

   (**d**) Based on your result from part **(c)**, what is the cycle frequency? Check your result in the limit of small oscillations.