Classical Mechanics

Preliminary Examination
Monday, August 24, 2015 9.00 AM to 12.00 Noon

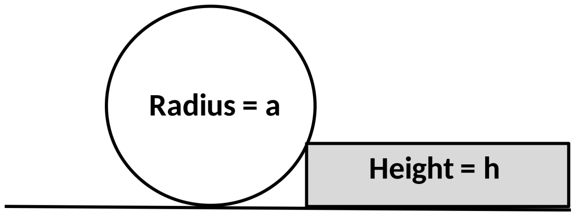
Answer a total of any THREE out of the four problems. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

Problem 1

A uniform ball of radius a rolling with velocity v on a level surface collides with a step height h < a as shown in the figure. Take the inelastic limit of the collision. Assume the coefficient of static friction between the sphere and the surface/step is sufficient to prevent slipping throughout the process.

a) Calculate the angular momentum about the center of mass of the ball

b) Find in terms of h and a, the minimum velocity needed for which the ball will jump over the step

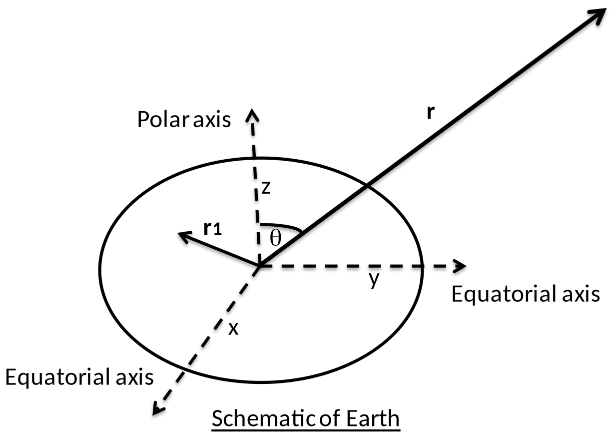


Problem 2

Due to polar flattening the earth has a slightly larger moment of inertia along the polar axis (C) than about the equatorial axis (A) i.e. C > A. Assume axial symmetry about the polar axis.

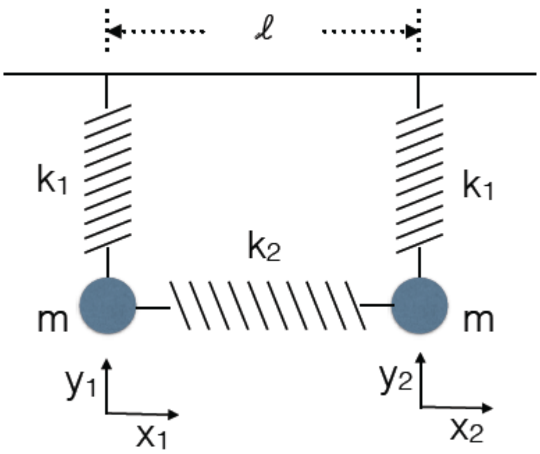
a) Expand the gravitational potential at a location **r** in a Taylor series. The small parameter for Taylor series here is ratio of magnitude of $\mathbf{r}_1$ and **r** where $\mathbf{r}_1$ is the integration variable over the interior of the earth and **r** is the location from the center of the earth.

b) Calculate the dominant terms (which varies as $1/r$ and $1/r^3$) of the gravitational potential far from the surface of the earth as a function of the angle θ between the vector **r** and polar axis (i.e. $z = r\cos\theta$)



Problem 3

You have a system of two masses $m_1 = m_2 = m$ and three springs, the vertical ones with a spring constant k_1 and a relaxed length (i.e., length when no forces act on it) of zero, the horizontal one with a spring constant k_2 and an equilibrium length ℓ. The two k_1 springs are fixed on the ceiling with a distance of ℓ between the fixing points. Assume the system to be two dimensional, with the coordinates as indicated in the figure. (Gravity is in the usual $-y$ direction.)



a) How many degrees of freedom does the system have? What are the kinetic and potential energies? What are the coordinates $x_{1,0}, x_{2,0}, y_{1,0}, y_{2,0}$ of the masses in the equilibrium?

b) Rewrite the Lagrangian such that it reads

$$\mathcal{L} = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j - \frac{1}{2} V_{ij} \eta_i \eta_j,$$

with $\eta = (\delta x_1, \delta x_2, \delta y_1, \delta y_2)$. For that, first write the Lagrangian in terms of the deviations from equilibrium, $\delta x_1 = x_1 - x_{1,0}$, etc., to second order in the η_i (Why can we ignore other orders?) Show the form of the 4×4 matrix T and show that

$$V = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k1 + k_2 & 0 & 0 \\ 0 & 0 & k_1 & 0 \\ 0 & 0 & 0 & k_1 \end{pmatrix}.$$

c) Find the eigenmodes for this system. How many of these modes are there?

Problem 4

Assume a general central potential $V(r)$ with total (constant) energy E and angular momentum L.

a) Show that the total energy of the system is

$$E = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r).$$

What is $V_{\text{eff}}(r)$? (Make sure to write this as a function where r is the only variable!)

b) For any stable circular orbit with $r = r_0$, what is $\frac{dV_{\text{eff}}}{dr}$? Show that in this case, the period of the system is

$$T = 2\pi \left(\frac{1}{mr} \frac{dV}{dr} \right)^{-\frac{1}{2}}.$$

Hint: Assume that the circular radius is an equilibrium radius.

c) Now assume that the orbit is slightly different from the circular orbit in part (a), $r(t) = r_0 + \epsilon(t)$, where at any time $|\epsilon(t)| \ll r_0$. Show that the form of this slight deviation can be written as a harmonic oscillator in the radius, oscillating with frequency ω_r. Show that

$$\omega_r^2 = \frac{1}{mr^3} \frac{d}{dr} r^3 \frac{dV}{dr}\bigg|_{r=r_0}.$$

d) For the Yukawa potential

$$V(r) = -\frac{GM}{r} e^{-kr},$$

show that oscillations around circular orbits only happen if

$$kr_0 < \frac{\sqrt{5}+1}{2}.$$

(Otherwise, the orbits would be unstable.)