Prelim and Course Exam: Classical Mechanics, Wednesday December 16, 2020. 6:00pm-9:00pm

For course and prelim credit answer the same **THREE** out of the four questions. If a student submits solutions to more than three problems, only the first three problems as listed on the exam will be graded. Students should write their solutions on blank 8.5 by 11 paper or in a blue book, putting their name on each page, the number of the problem and the number of the page in their solution on each page (i.e. 2-1 means first page of problem 2). Also each problem solution should be on a separate set of pages (i.e. not putting parts of two different problems on the same page). At the end of the exam students should scan in their solutions in sequence using a cell phone or a scanner and email them in a file or files (ideally pdf) to the prelim committee chair philip.mannheim@uconn.edu no later than 15 minutes after the end time of the exam. (It might be easier to transfer the files to a laptop first.) Label both the email header and the file or files with your name and the name of the exam. In the email state which problems you have attempted and state how many pages there are for each of the problems. The chair will immediately check if the emailing is readable or if a resend is required.

1. A plausible model of a ring molecule consists of three masses $2m$, m and m which can slide on a fixed horizontal circular wire of radius R. Assume there is no friction between the masses and the wire and they are connected by three identical light springs of spring constant k, coiled around the same circle. The (small) angles ϕ_1, ϕ_2 and ϕ_3 corresponding to the three masses are measured from their equilibrium positions – see Fig. 1. (The springs are under no tension when in equilibrium.)
   (a) Write down a Lagrangian for small oscillations in terms of the above angles and their time derivatives.
   (b) Find the normal frequencies for small oscillations.
   (c) Obtain the corresponding normal modes and describe them.

2. A uniform rod of mass M and length l has a small hook of mass m at one end that loops over a fixed wire. The hook is free to slide smoothly on this straight, horizontal wire. At time $t = 0$, the rod is held parallel to the wire and let go. The rod is constrained to move and rotate in the vertical plane containing the wire (see Fig. 2).

   How many degrees of freedom are there in this problem?
   (a) For the subsequent motion under gravity, write down a Lagrangian.
   (b) At time t, when the rod is inclined to the wire at an angle θ, the hook would have slipped a distance x along the wire. Write down equations of motion and find an expression for x in terms of θ and constants you identify. Specify any underlying law(s) of physics here.
   (c) Use the work-energy theorem to obtain an expression for $\dot{\theta}$ in terms of θ and constants.

3. (a) Starting with the relativistic Lagrangian

$$L = -mc^2\sqrt{1 - v^2/c^2} - V(r)$$

   for a particle of mass m moving with speed v under a central potential $V(r)$ in a plane, find the generalized momenta p_r and p_ϕ where r and ϕ are the (plane) polar coordinates of the particle with respect to a fixed origin. Identify a cyclic coordinate here, if any. Show that the Hamiltonian is given by $H = mc^2/\sqrt{1 - v^2/c^2} + V(r)$ and express it in terms of p_r, p_ϕ and V (c is the speed of light in a vacuum).
   (b) Using the fundamental Poisson brackets, find the values of the constants α and β for which the equations

$$Q = q^\alpha \cos(\beta p), \quad P = q^\alpha \sin(\beta p)$$

   represent a canonical transformation.

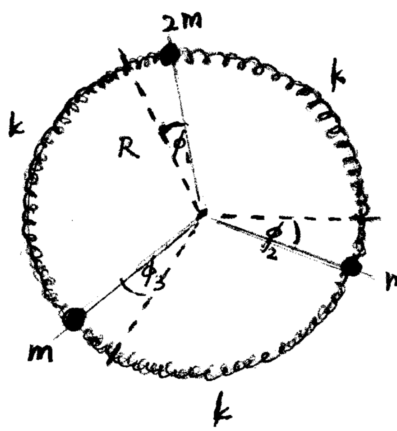
4. A dumbbell consisting of two point masses connected by a light rod of length $2a$ rotates with angular velocity $\vec{\omega}$ about a vertical axis through a fixed origin O (midpoint of the rod), keeping the angle α (between the vertical and the rod) constant (see Fig. 3). Neglect gravity in this problem.

(a) Using a figure, clearly identify a set of principal axes for this problem and write down the principal moments of inertia by inspection.

(b) Suppose $\vec{\omega}$ is held constant in the lab frame. Using Euler's equations of motion or otherwise, express the components of the necessary torque along the principal axes in terms of $\omega = |\vec{\omega}|$ and other variables you identify.

(c) Find the angular momentum and kinetic energy of the dumbbell and explain why the kinetic energy stays constant in time in the presence of a nonzero torque.

Fig. 1



[DASHED LINES POINT TO EQUILIBRIUM POSITIONS OF THE PARTICLES $\phi_1 = \phi_2 = \phi_3 = 0$]

Fig. 2

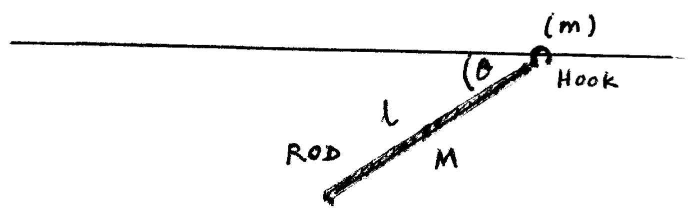


Fig. 3