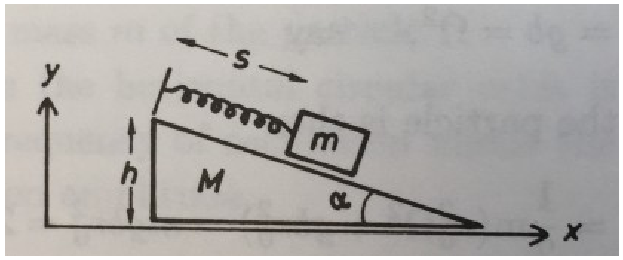
Preliminary Exam: Classical Mechanics, Monday January 13, 2020. 9:00-12:00

Answer a total of any **THREE** out of the four questions. Put the solution to each problem in a **SEPARATE** blue book and put the number of the problem and your name on the front of each book. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.



1. A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface, as in the figure.

   (**a**) Given that the relaxed length of the spring is d, find the length of the spring s_0 when both the block and the wedge are at rest.

   (**b**) Find the Lagrangian for the system as a function of the x coordinate of the wedge and the length s of the spring.

   (**c**) Find the eigenmodes and eigenfrequencies of the system.

   (**d**) What are the eigenfrequencies in the limit $\alpha = \pi/2$ and $M = \infty$.

2. Three identical objects, each of mass of m, are connected by springs of spring constant k as shown in the figure. In equilibrium the distance AB and the distance BC are each equal to a. The motion is confined to one dimension. At $t = 0$ the masses are at rest at their equilibrium positions. Starting at $t = 0$ mass A is subjected to an external force

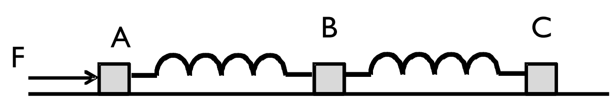
$$F = f\cos(\omega t).$$

   (**a**) Determine the equations of motion of A, B and C.

   (**b**) Determine the normal modes and the corresponding frequencies of the system.

   (**c**) Write down the equation of motion of each of the normal modes in the presence of the force F.

   (**d**) Calculate the position and velocity of mass C using the initial conditions at $t = 0$: $x_A(t = 0) = 0$, $x_B(t = 0) = a$, $x_C(t = 0) = 2a$, $v_A(t = 0) = 0$, $v_B(t = 0) = 0$, $v_C(t = 0) = 0$, where x_A, x_B and x_C are the respective positions of masses A, B and C and v_A, v_B and v_C are their respective velocities.



3. Assume you have a central potential of $V(r) = -k\,e^{-\frac{r}{r_0}}$, where k and r_0 are positive constants.

   a) Using the equivalent one-dimensional problem, show that there exists a stable circular orbit if and only if

   $$\frac{L^2}{m} < 27\,k\,r_0^2\,e^{-3},$$

   where L is the angular momentum and m the (reduced) mass. For this case, make a rough sketch of the effective potential $\tilde{V}(r) = V(r) + \frac{L^2}{2mr^2}$. (*Hint: In order to find the transition point between the existence or not of a minimum, find the point where both the first and second derivative of $\tilde{V}(r)$ are zero. Explain why this is the transition point.*) Show that for the circular orbit radius, one gets

   $$\frac{d^2}{dr^2}\tilde{V}(r) = \frac{L^2}{mr^4}\left(3 - \frac{r}{r_0}\right).$$

   b) Assume you have an energy E slightly above that of a circular orbit of radius R, such that (to lowest order) $E \approx m\delta\dot{r}^2/2 + \tilde{V}(R) + \delta r^2 \tilde{V}''(R)/2$ with $r = R + \delta r$. Remember that the angle $\Delta\phi$ between the turning points for r (i.e., between a maximum r_{max} and minimum r_{min} radius) is given by

   $$\Delta\phi = \frac{L}{\sqrt{2m}} \int_{r_{min}}^{r_{max}} \frac{dr}{r^2\sqrt{E - \tilde{V}(r)}}.$$

   Show that

   $$\Delta\phi = \frac{\pi}{\sqrt{3 - \frac{R}{r_0}}}$$

You might find the following integral useful:

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi.$$

4. A solid constant-density spherical planet of mass M_0 rotates with angular velocity ω_0. Suddenly, a (pointlike) asteroid of mass αM_0 ($\alpha \ll 1$) smashes into and sticks to the surface of the planet at a location which is at polar angle θ relative to the initial rotational axis. The new mass distribution is no longer spherically symmetric, and the rotational axis will precess. Recall Euler's equation

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{N}_{\text{ext}}$$

for rotations in a body-fixed frame.

   a) Draw a sketch of the rigid body after the collision using body-centered coordinates of your choice. Your diagram should show the body axes (numbered), the angular momentum vector $\vec{L}$, and the approximate angular velocity vector $\vec{\omega}$ immediately after impact. You may assume that the momentum of the asteroid was too small to appreciably affect the rotation speed or total angular momentum of the planet during the collision.

   b) What is the new inertia tensor $I_{\alpha\beta}$ along the principle center-of-mass frame axes? Do not forget that the center of mass is no longer at the center of the sphere! Recall $I = \frac{2}{5}MR^2$ for a solid sphere.

   b) What is the period of precession of the rotational axis in terms of the original length of the day $\tau = 2\pi/\omega_0$? (How does the Euler"s equation for the three components now look, when it is written in terms of I and $\vec{\omega}$?)