Preliminary Exam: Classical Mechanics, Monday, January 14, 2019, 9am-noon

Answer a total **THREE** questions out of **FOUR**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

1. A simple pendulum of mass m and length l hangs from a trolley with total mass M running on smooth horizontal rails. Ignore the moment of inertia of the wheels. The pendulum swings in a plane parallel to the rails; see Figure 1.

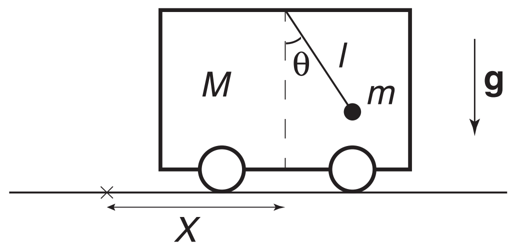

Figure 1.

(a) Using the position of the trolley X and the angle of inclination θ as your generalized coordinates, write down the Lagrangian and Lagrange's equations.

(b) Suppose the whole setup is first at rest, with the pendulum held at the angle θ_0, and the pendulum is then released. What is the angular velocity of the pendulum when it points straight down?

Hint: One way to solve this problem is to find two constants of the motion.

2. (a) Consider a free particle (mass m, charge q) in a constant (in space and time) magnetic field $\mathbf{B}$. Show that the Lagrangian may cast in the form

$$L = \tfrac{1}{2} m \mathbf{v} \cdot \mathbf{v} + \tfrac{1}{2} q [\mathbf{B} \cdot (\mathbf{r} \times \mathbf{v})] \,.$$

(b) If you transform to a coordinate system rotating at the angular velocity $\boldsymbol{\omega}$, the kinematics implies that the velocity vector itself changes. Write down the well-known relation between the lab frame velocity $\mathbf{v}$ and rotating-frame velocity $\mathbf{V}$.

(c) When a constant magnetic field $\mathbf{B}$ is present, it is possible to find a rotating frame in which the particle is free in the direction of $\mathbf{B}$, and behaves like an isotropic two-dimensional harmonic oscillator in the directions perpendicular to $\mathbf{B}$. Find the corresponding angular velocity $\boldsymbol{\omega}$, and the angular frequency of the harmonic oscillator.

3. Imagine the system shown in Figure 2: a pointlike mass m, hanging on a string. The string is fixed on a spring with spring constant k on the other end. The spring, with zero equilibrium length (i.e., the length that it would have if lying unattached on a table), is attached to the wall. The string of length L is hanging over a small metal peg at a distance $L/2$ from the wall (small circle in the figure) with no friction. The system exists in normal gravity. Assume that the string is taut at all times.

   (a) Treating this as a 2D problem (as in the figure), find the Lagrangian and the Lagrange equations of motion.

   (b) There is an equilibrium point. What is it? Show that small oscillations around this point give independent oscillations of ϕ and r. What are the frequencies?

   (c) If $k/m \gg g/L$, show that the initial condition in the right side of Figure 2 leads to an approximately periodic motion. Show that the period of the ϕ-part is given by

$$\Delta t = \sqrt{\frac{L}{g}} \underbrace{\int_0^{\pi} \frac{d\phi}{\sqrt{\sin\phi}}}_{\approx 5.24} .$$

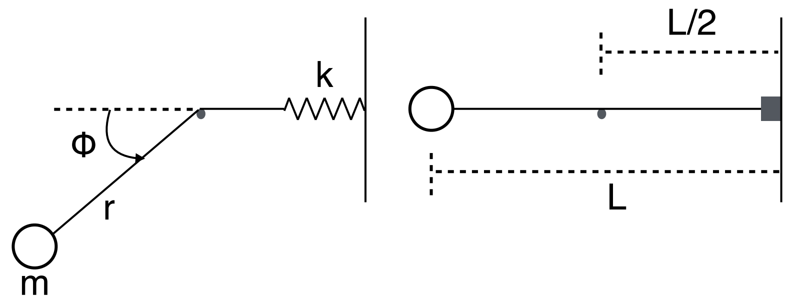

Figure 2. System from Problem 3. On the left side, the system while moving. The right side shows the starting point for part (c) with a completely unextended spring and the whole string parallel to the floor. Gravity goes downward.

4. A central potential in a 3D (Euclidean) space is given by

$$U = \begin{cases} 0 & \text{for} \quad r \le r_1 \\ \epsilon > 0 & \text{for} \quad r_1 < r \le r_2 \\ \infty & \text{for} \quad r_2 < r \end{cases}$$

   (a) For a point mass m moving in this potential, write the equivalent one-dimensional problem (with variable r). What types of curves does the mass follow for $r \neq r_1, r_2$?

   (b) Show that for $0 \le E < \epsilon$ (with total energy E), one possible periodic orbit is a square. What, in this case, is the relationship between energy E and angular momentum ℓ?

   (c) Are there circular orbits? Which? If yes, what is the relationship between E and ℓ?

   (d) What happens if, in the case of $E > \epsilon$, the particle moves "over the edge" from $r < r_1$ to $r > r_1$? Thus, sketch what types of orbits can happen in this energy regime.