Preliminary Exam: Classical Mechanics, Monday January 9, 2017. 9:00-12:00

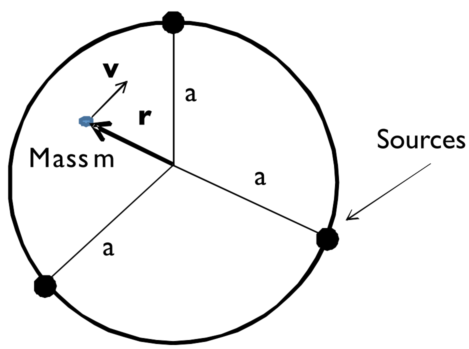
Answer a total of any **THREE** out of the four questions. For your answers you can use either the blue books or individual sheets of paper. If you use the blue books, put the solution to each problem in a separate book and put the number of the problem on the front of the book. If you use the sheets of paper, use different sets of sheets for each problem and sequentially number each page of each set. Be sure to put your name on each book and on each sheet of paper that you submit. If you submit solutions to more than three problems, only the first three problems as listed on the exam will be graded.

1. Three fixed point sources are equally spaced about the circumference of a circle of radius a. The force exerted by each source on a point mass m is attractive and is given by $\mathbf{F} = -k\mathbf{R}$ where $\mathbf{R}$ is the vector drawn from the source to the point mass. The point mass is free to move in all three dimensions, and is placed in the force field at $t = 0$ with initial conditions $\mathbf{r} = \mathbf{r}_0$ and velocity $\mathbf{v}_0$. For this problem do not include the effect of gravity.

   (**a**) Using suitable coordinates, write an expression for the force acting on the mass.

   (**b**) Solve the equation of motion of the mass, and find $\mathbf{r}(t)$ as a function of k, $\mathbf{r}_0$ and $\mathbf{v}_0$.

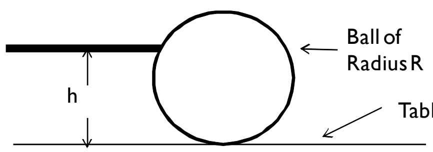
   (**c**) Under what circumstances does $\mathbf{r}(t)$ correspond to a circular orbit?

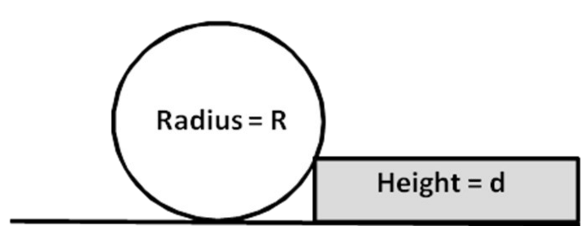


2. A uniform density billiard ball of radius R and mass M is struck with a horizontal cue stick at a height h above a horizontal billiard table. The moment of inertia I of a uniform density sphere is $I = 2MR^2/5$ where M is the mass of the sphere and R is its radius. The following three points all lie in a common vertical plane: the center of the ball, the point of contact between the ball and the cue stick, and the point of contact between the ball and the table. Take the gravity of the earth to produce a uniform acceleration g acting in the downward vertical.

   (**a**) Find the value of h for which the ball will roll without slipping no matter how small the coefficient of friction between the ball and the table might be.

   (**b**) Consider the case where the ball is rolling with a velocity $\mathbf{v}$ on a level horizontal surface and collides with a step of height $d < R$ as shown in the figure. Take the inelastic limit of the collision. Assume that the coefficient of static friction between the ball and both the step and the surface is sufficient to prevent slipping throughout the process. For a given d and R determine the minimum velocity needed for which the ball will be able to jump over the step.





3. Consider a different (not our!) world with different (not Newton's) equations of motion which contain higher (than second order) time derivatives of the positions. In such a world the Lagrange function would not only depend on positions $q = q(t)$ and velocities $\dot{q} = dq(t)/dt$, but also on higher-order derivatives of the positions.

(a) Consider a general one-dimensional system described by $L = L(\ddot{q}, \dot{q}, q)$. By extremizing the action $S = \int_{t_1}^{t_2} \mathrm{d}t\, L$ (with positions and velocities fixed at the end points) derive the equations of motion

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0\,.$$

(b) Consider the specific system

$$L(\ddot{q}, \dot{q}, q) = -\frac{1}{2}\, m\, \ddot{q}\, q - \frac{1}{2}\, k\, q^2$$

Derive the equation of motion, and solve it for appropriate initial conditions. Which system is this?

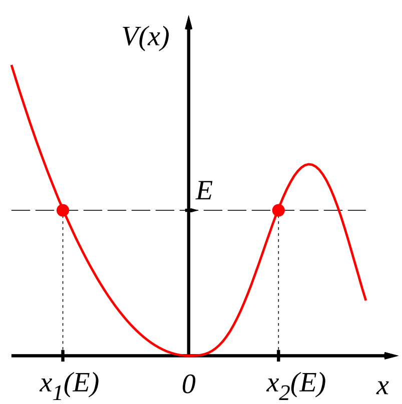
(c) What is special about the particular Lagrangian in part (**b**) that makes it reproduce a familiar equation of motion well known from Newtonian physics?

(d) What kind of term should $L(\ddot{q}, \dot{q}, q)$ contain in order to describe a system which definitely does not correspond to the classical mechanics in our world? (It is sufficient to provide one example.)

4. Consider the one-dimensional motion of a particle of mass m with energy $E > 0$ in the general potential $V(x)$ with the properties and generic shape sketched in the figure.

(a) Derive for a general $V(x)$ the formula for the period $T = T(E)$ for the particle's motion from the point x_1 to x_2 and back again as function of the particle's energy E.

(b) Determine T for the specific potential $V(x) = A\,|x|^N$ with $N > 0$. For which value of N is the period T independent of the energy E?



Remarks:
1. You can make use of the quoted results in later parts of the problems, even if you were not able to derive them.
2. Possibly useful integral

$$I(\alpha) \equiv \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^\alpha}} = \sqrt{\pi}\, \frac{\Gamma(1 + \frac{1}{\alpha})}{\Gamma(\frac{1}{2} + \frac{1}{\alpha})}\,, \quad \mathrm{Re}\,\alpha > 0$$