CLASSICAL MECHANICS

Preliminary Examination

Monday January 9, 2012

09:00 - 12:00 in P-121

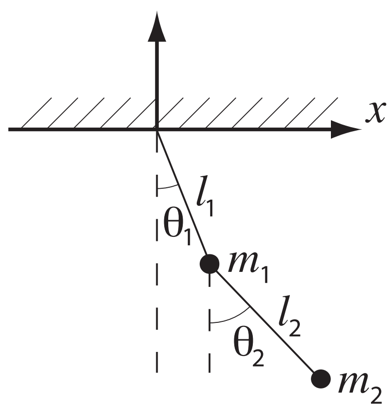
Answer a total of **THREE** questions. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book or on individual sheets of paper. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

Problem 1. (a) Obtain the Lagrangian of the double pendulum illustrated in the figure below. The masses are constrained to move in a vertical plane, and the rods between the masses are considered to be massless.

(b) Write the Lagrangian of part (a) in the limit of small angular displacements from equilibrium, $\theta_1 \ll 1$ and $\theta_2 \ll 1$, neglecting terms higher than second order in the quantities θ_1 and θ_2, with $\cos\theta \simeq 1 - \theta^2/2$, $\sin\theta \simeq \theta$, and obtain Lagrange's equations of motion for θ_1 and θ_2.

(c) Assuming solutions of the form $\theta_k = A_k e^{-i\omega t}$, find the normal frequencies of the system.



Problem 2. In this astrophysics exercise denote the mass, radius and rotation angular velocity of the Earth by M_e, R_e and ω_e. A small asteroid with a mass $m \ll M_e$ strikes the assumedly perfectly spherical Earth at the co-latitude (polar angle) θ. Assume that the sole effect of the asteroid is to deposit its mass at the point of impact. This breaks the full rotation symmetry of the Earth, and thus fixes the direction for one principal axis of rotation.

(a) Show that the new moments of inertia are $I_1 = I_2 = \frac{2}{5} M_e R_e^2 + m R_e^2 + \mathcal{O}(m^2)$, $I_3 = \frac{2}{5} M_e R_e^2$.

The result is that the axis of the Earth starts precessing about an axis that goes through the center of the Earth and the point of the impact.

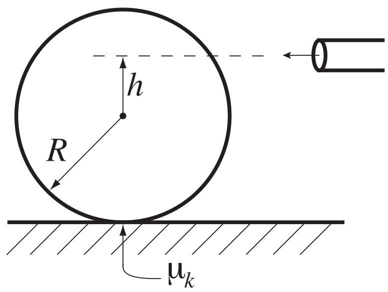
(b) Find the angular frequency of the precession.

Recall that Euler's equations without applied torque read $I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2)$ and so forth, with cyclic permutations of the directions 1, 2 and 3.

Problem 3. Consider a spherical billiard ball of uniform density with mass M and radius R, so that the moment of inertia is $\frac{2}{5}MR^2$. This ball is initially at rest on a surface with which it has a coefficient of kinetic friction μ_k. The ball is struck with a thin stick (pool cue) and given a horizontal impulse. The point of contact is a distance h above the center of the ball. Immediately after the impact, the ball has a speed v_0; eventually, when the ball is rolling without slipping, it acquires a maximum speed of $\frac{9}{7}v_0$.

(a) From this information, determine the value of h.

(b) If the ball's total kinetic energy after the initial impact is K_0, what fraction of the kinetic energy is lost by the time the ball rolls without slipping?



Problem 4. A particle is attracted to a force center by a force that varies as $F = -k/r^3$.

(a) Using the well-known Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$, where $V(r)$ is the potential energy, derive Lagrange's equations of motion for the radial distance r and the polar angle θ.

(b) Using the equations of motion in part (a), and defining $u \equiv 1/r$, derive the equation for the orbit of the particle

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{mk}{J^2}\right)u = 0\,.$$

Here $J = mr^2\dot{\theta}$ is the value of the conserved angular momentum.

(c) Solve the equation of part (b) for $u(\theta)$ both with $J^2 > mk$ and $J^2 < mk$, and discuss how the nature of the orbits differ.