Disconnected quark loop contribution to Hadronic Light-by-light diagram

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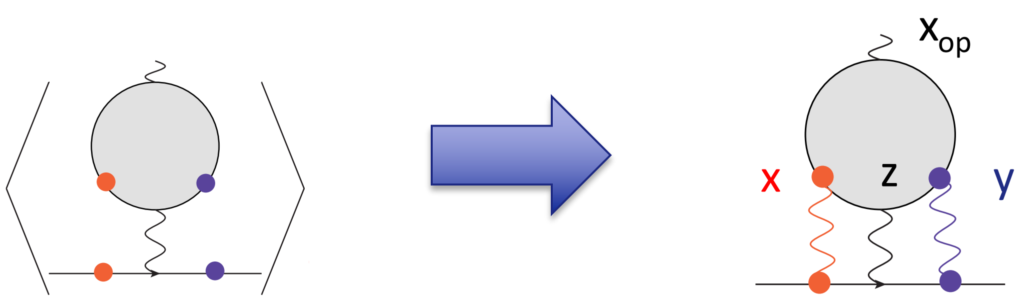
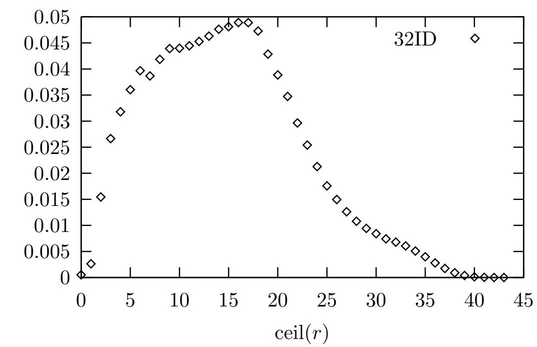
Coordinate space Point photon method

[Luchang Jin et al. , PRD93, 014503 (2016)]

- Treat all 3 photon propagators exactly (3 analytical photons) , which makes the quark loop and the lepton line connected :
  disconnected problem in Lattice QED+QCD -> connected problem with analytic photon
- QED 2-loop in coordinate space. Stochastically sample, two of quark-photon vertex location x,y, z and x_{op} is summed over space-time exactly





- Short separations, Min[|x-z|,|y-z|,|x-y|] < R ~ O(0.5) fm, which has a large contribution due to confinement, are summed for all pairs
- longer separations, Min[|x-z|,|y-z|,|x-y|] >= R, are done stochastically with a probability shown above (Adaptive Monte Carlo sampling)

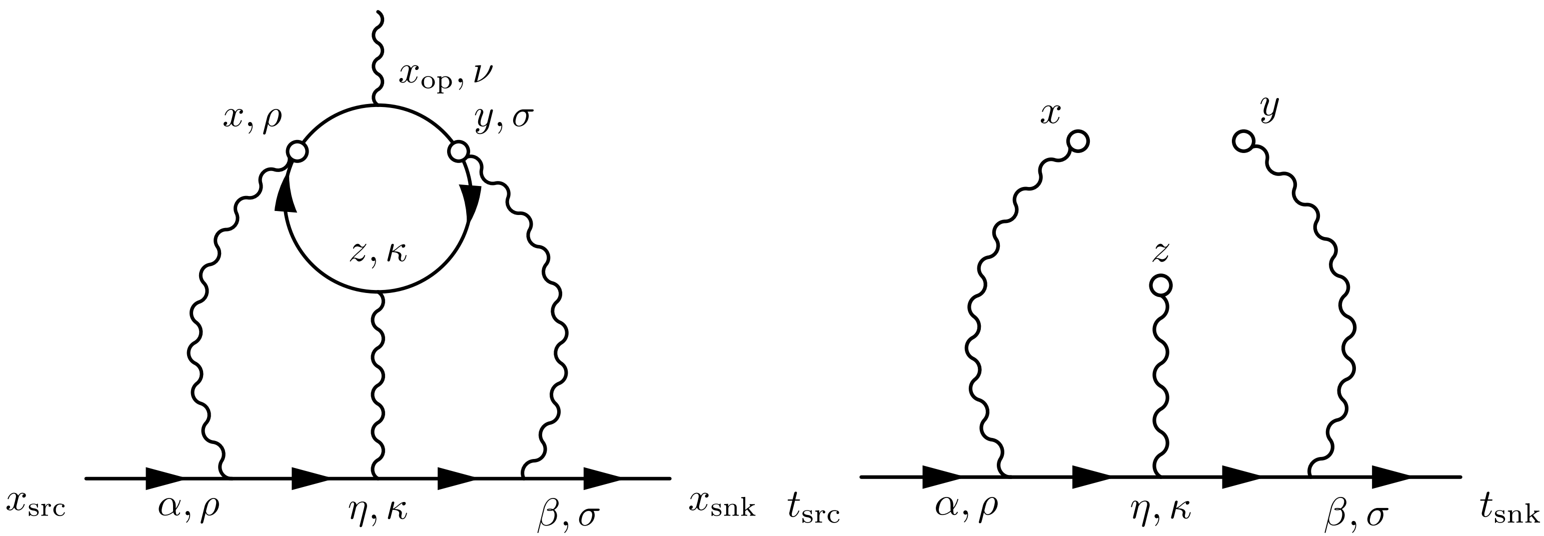
HLbL point source method [L. Jin et al. 1510.07100]

- Anomalous magnetic moment, $F_2(q^2)$ at $q^2 \to 0$ limit

$$\frac{F_2^{\text{cHLbL}}(q^2=0)}{m}\frac{(\sigma_{s',s})_i}{2} = \frac{\sum_{x,y,z,x_{\text{op}}}}{2VT}\epsilon_{i,j,k}\left(x_{\text{op}}-x_{\text{ref}}\right)_j \cdot i\bar{u}_{s'}(\vec{0})\mathcal{F}_k^C\left(x,y,z,x_{\text{op}}\right)u_s(\vec{0})$$

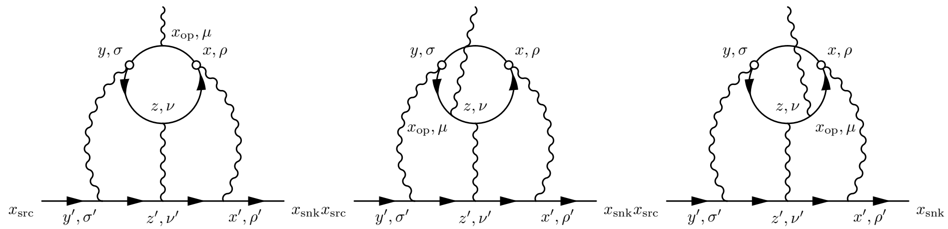
- Stochastic sampling of x and y point pairs. Sum over x and z.

$$\mathcal{F}_\nu^C\left(x,y,z,x_{\text{op}}\right) = (-ie)^6\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\text{op}}),$$



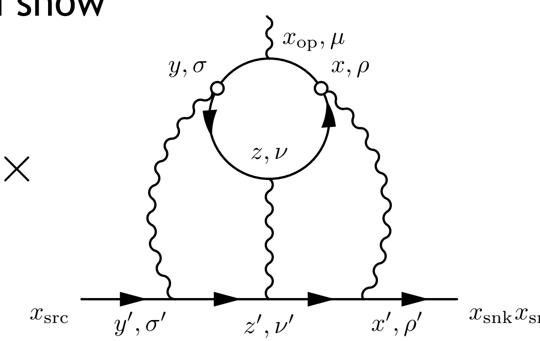
Conserved current & moment method

- [conserved current method at finite q2] To tame UV divergence, one of quark-photon vertex (external current) is set to be conserved current (other three are local currents). All possible insertion are made to realize conservation of external currents config-by-config.



- [moment method , q2→0] By exploiting the translational covariance for fixed external momentum of lepton and external EM field, q->0 limit value is directly computed via the first moment of the relative coordinate, xop – (x+y)/2, one could show

$$\frac{\partial}{\partial q_i}\mathcal{M}_\nu(\vec{q})|_{\vec{q}=0} = i \sum_{x,y,z,x_{\rm op}} (x_{\rm op} - (x+y)/2)_i \times$$



to directly get $F_2(0)$ without extrapolation.

$$\text{Form factor}: \ \Gamma_\mu(q) = \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2\, m_l} F_2(q^2)$$

Simulation details [RBC/UKQCD 2015]

two gauge field ensembles generated by RBC/UKQCD collaborations

Domain wall fermions: chiral symmetry at finite a

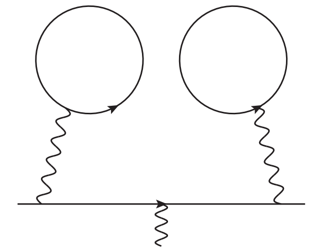
Iwasaki Gauge action (gluons)

- pion mass $m_\pi = 139.2(2)$ and 139.3(3) MeV ($m_\pi L \lesssim 4$)
- lattice spacings $a = 0.114$ and 0.086 fm
- lattice scale $a^{-1} = 1.730$ and 2.359 GeV (+1.0 GeV, +1.38 GeV)
- lattice size $L/a = 48$ and 64
- lattice volume $(5.476)^3$ and $(5.354)^3$ fm^3 (+7 fm + 9.6 fm)

Use all-mode-average (AMA) [Blum et al 2012] and low-mode- averaging (LMA) [Giusti et al, 2004, Degrand et al 2005, Lehner 2016 for HVP] techniques for improved statistics by more than three orders of magnitudes compared to basic CG, and $\times 10$ smaller memory via multigrid-Lanczos [Jung Lehner 2017]

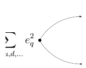
HVP
disconnected quark loop contribution

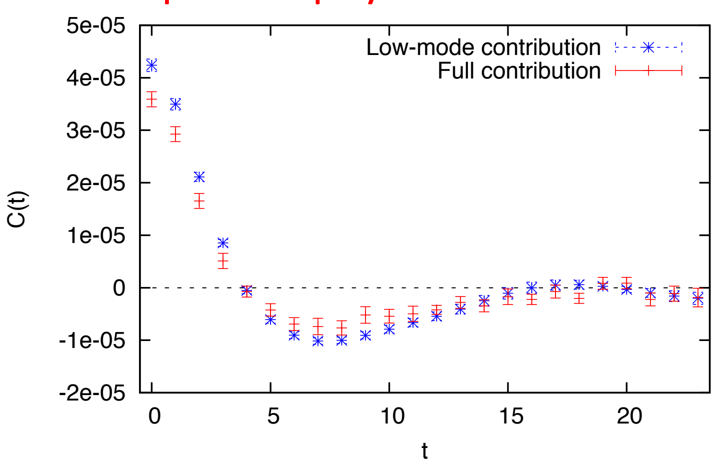
- [C. Lehner et al. (RBC/UKQCD 2015, arXiv:1512.09054, PRL)]
- Very challenging calculation due to statistical noise
- Small contribution, vanishes in SU(3) limit,
  Qu+Qd+Qs = 0
- Use low mode of quark propagator, treat it exactly
  (all-to-all propagator with sparse random source)
- First non-zero signal

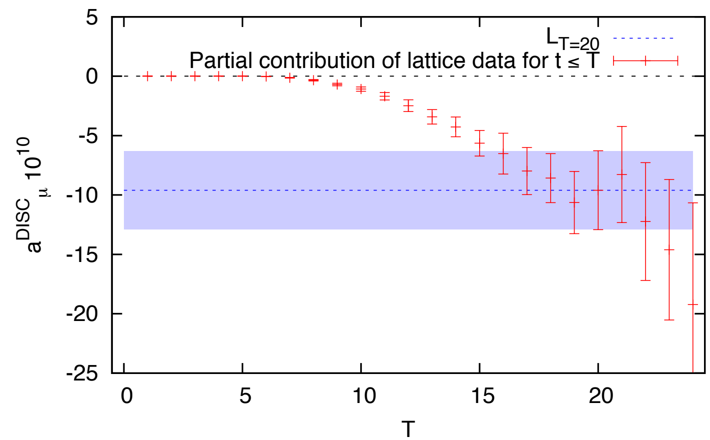


$$a_\mu^{\mathrm{HVP\ (LO)\ DISC}} = -9.6(3.3)_{\mathrm{stat}}(2.3)_{\mathrm{sys}} \times 10^{-10}$$

Sensitive to m_π
crucial to compute at physical mass







HVP

Current conservation & subtractions

- conservation => transverse tensor

$$\Pi^{\mu\nu}(q) = (\hat{q}^2\delta^{\mu\nu} - \hat{q}^\mu\hat{q}^\nu)\Pi(\hat{q}^2)$$

- In infinite volume, q=0, $\Pi_{\mu\nu}(q) = 0$
- For finite volume, $\Pi_{\mu\nu}(0)$ is exponentially small (L.Jin, use also in HLbL)

$$\int_V dx^4 \langle V_\mu(x)\mathcal{O}(0)\rangle = \int_V dx^4\, \partial_x \left(x\langle V_\mu(x)\mathcal{O}(0)\rangle\right)$$

$$= \int_{\partial V} dx^3\, x\langle V_\mu(x)\mathcal{O}(0)\rangle \propto L^4 \exp(-ML/2) \to 0$$

- e.g. DWF L=2, 3, 5 fm $\Pi_{\mu\nu}(0)$ = 8(3)e-4, 2(13)e-5, -1(5)e-8
- Subtract $\Pi_{\mu\nu}(0)$ alternates FVE, and reduce stat error "-1" subtraction trick [Bernecker & Meyer, Maintz] :

$$\Pi^{\mu\nu}(q) - \Pi^{\mu\nu}(0) = \int d^4x (e^{iqx} - 1)\langle J^\mu(x) J^\nu(0)\rangle$$

cHLbL Subtraction using current conservation

- From current conservation, $\partial_\rho V_\rho(x) = 0$, and mass gap, $\langle x V_\rho(x) \mathcal{O}(0) \rangle \sim |x|^n \exp(-m_\pi |x|)$

$$\sum_x \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) = \sum_x \langle V_\rho(x) V_\sigma(y) V_\kappa(z) V_\nu(x_{\sf op}) \rangle = 0$$

$$\sum_z \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op}) = 0$$

at $V \to \infty$ and $a \to 0$ limit (we use local currents).

- We could further change QED weight

$$\mathfrak{G}^{(2)}_{\rho,\sigma,\kappa}(x, y, z) = \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x, y, z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y, y, z) - \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(x, y, y) + \mathfrak{G}^{(1)}_{\rho,\sigma,\kappa}(y, y, y)$$

without changing sum $\sum_{x,y,z} \mathfrak{G}_{\rho,\sigma,\kappa}(x, y, z) \mathcal{H}^C_{\rho,\sigma,\kappa,\nu}(x, y, z, x_{\rm op})$.

- Subtraction changes discretization error and finite volume error.
- Similar subtraction is used for HVP case in TMR kernel, which makes FV error smaller.
- Also now $\mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(z, z, x) = \mathfrak{G}^{(2)}_{\sigma,\kappa,\rho}(y, z, z) = 0$, so short distance $\mathcal{O}(a^2)$ is suppressed.
- The 4 dimensional integral is calculated numerically with the CUBA library cubature rules. (x, y, z) is represented by 5 parameters, compute on N^5 grid points and interpolates. ($|x - y| < 11$ fm).

Dramatic Improvement !

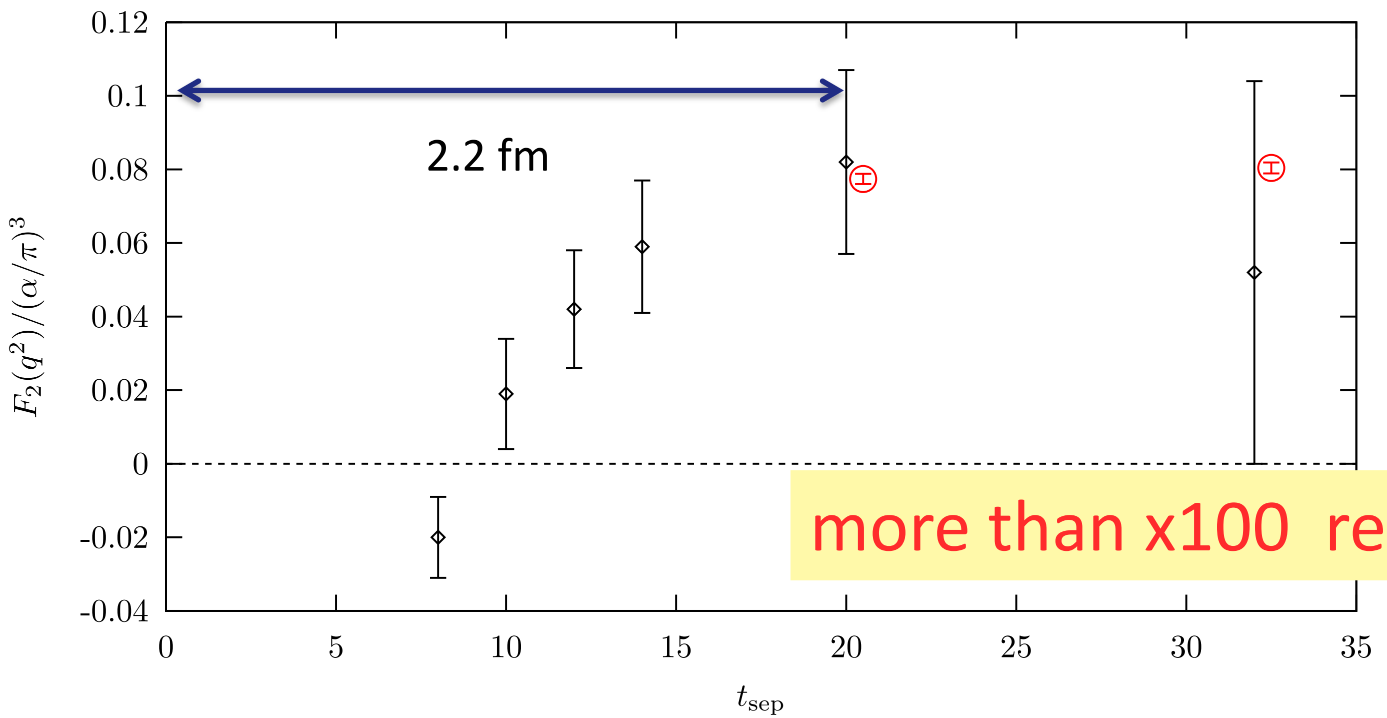
Luchang Jin

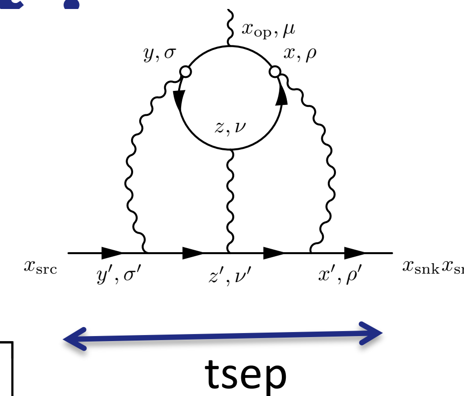
a=0.11 fm, 24^3x64 $(2.7\text{ fm})^3$,

m_π = 329 MeV, m_μ =~ 190 MeV, e=1

$q = 2\pi/L$ $N_{\rm prop} = 81000$

$q = 0$ $N_{\rm prop} = 26568$





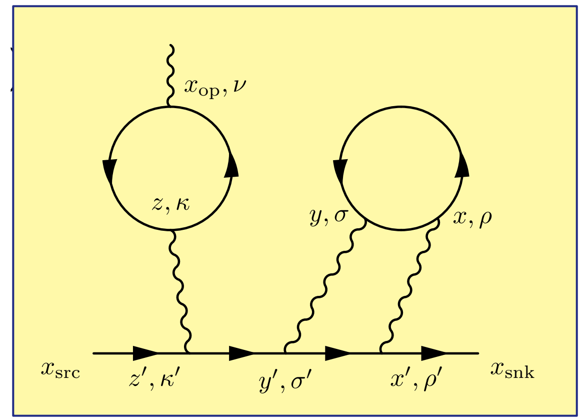
more than x100 reduced cost !

Method	$F_2/(\alpha/\pi)^3$	$N_{\rm conf}$	$N_{\rm prop}$	$\sqrt{\rm Var}$
Conserved	0.0825(32)	12	$(118 + 128) \times 2 \times 7$	0.65
Mom.	0.0804(15)	18	$(118 + 128) \times 2 \times 3$	0.24

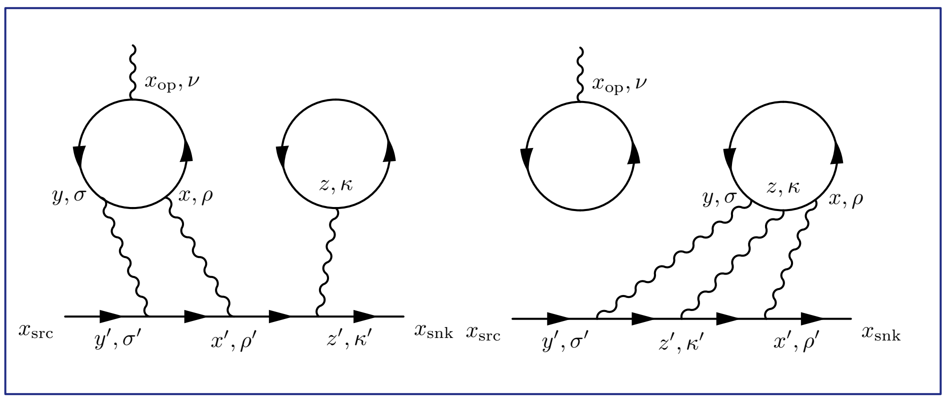
SU(3) hierarchies for d-HLbL

- At $m_s = m_{ud}$ limit, following type of disconnected HLbL diagrams
  $Q_u + Q_d + Q_s = 0$ [Mainz, Lehenr for HVP]
- Other diagrams suppressed by
  $O(m_s - m_{ud})/3$, $O((m_s - m_{ud})^2)$, and $O((m_s - m_{ud})^3)$

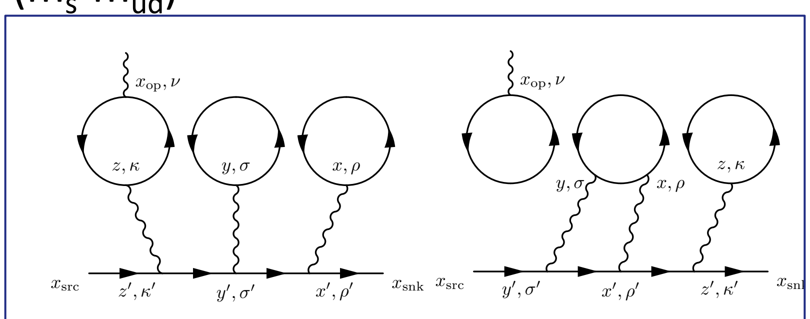
$(m_s - m_{ud})^0$



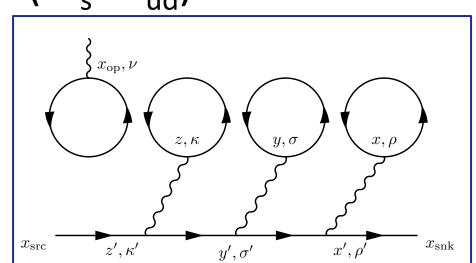
$(m_s - m_{ud})/3$



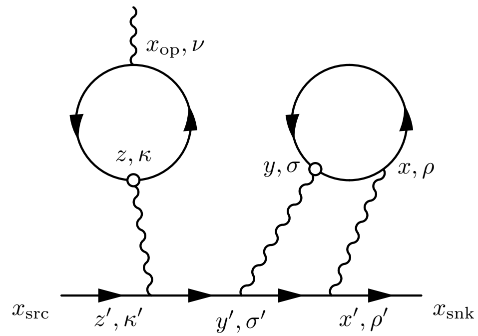
$(m_s - m_{ud})^2$



$(m_s - m_{ud})^3$



Disconnected calculation



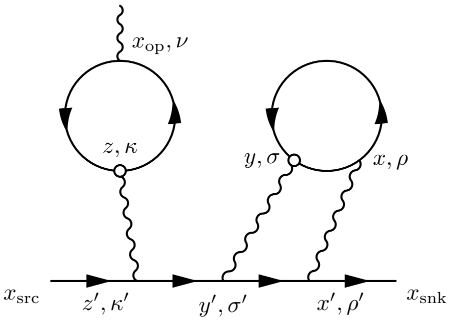
- We can use two point source photons at y and z, which are chosen randomly. The points $x_{\rm op}$ and x are summed over exactly on lattice.
- Only point source quark propagators are needed. We compute M point source propagators and all M^2 combinations of them are used to perform the stochastic sum over $r = z - y$.

$$\mathcal{F}^D_\nu(x,y,z,x_{\rm op}) = (-ie)^6\,\mathcal{G}_{\rho,\sigma,\kappa}(x,y,z)\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) \quad (13)$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = \left\langle \frac{1}{2}\Pi_{\nu,\kappa}(x_{\rm op},z)\left[\Pi_{\rho,\sigma}(x,y) - \Pi^{\rm avg}_{\rho,\sigma}(x-y)\right]\right\rangle_{\rm QCD} \quad (14)$$

$$\Pi_{\rho,\sigma}(x,y) = -\sum_q (e_q/e)^2\,{\rm Tr}[\gamma_\rho S_q(x,y)\,\gamma_\sigma S_q(y,x)]. \quad (15)$$

Disconnected claculation



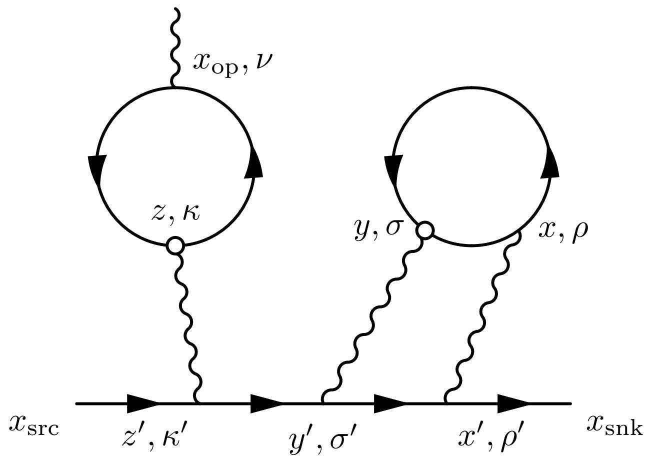
$$\frac{F_2^{\rm dHLbL}(0)}{m}\frac{(\sigma_{s',s})_i}{2} = \sum_{r,x}\sum_{x_{\rm op}} \frac{1}{2}\epsilon_{i,j,k}(\tilde{x}_{\rm op})_j \cdot i\,\bar{u}_{s'}(\vec{0})\,\mathcal{F}_k^D(x, y=r, z=0, x_{\rm op})u_s(\vec{0}) \quad (16)$$

$$\mathcal{H}^D_{\rho,\sigma,\kappa,\nu}(x,y,z,x_{\rm op}) = \left\langle \frac{1}{2}\Pi_{\nu,\kappa}(x_{\rm op},z)\,[\Pi_{\rho,\sigma}(x,y) - \Pi^{\rm avg}_{\rho,\sigma}(x-y)] \right\rangle_{\rm QCD} \quad (17)$$

$$\sum_{x_{\rm op}} \frac{1}{2}\epsilon_{i,j,k}(x_{\rm op})_j\,\langle \Pi_{\rho,\sigma}(x_{\rm op},0)\rangle_{\rm QCD} = \sum_{x_{\rm op}} \frac{1}{2}\epsilon_{i,j,k}(-x_{\rm op})_j\,\langle \Pi_{\rho,\sigma}(-x_{\rm op},0)\rangle_{\rm QCD} = 0$$

- Because of the parity symmetry, the expectation value for the left loop average to zero.
- $[\Pi_{\rho,\sigma}(x,y) - \Pi^{\rm avg}_{\rho,\sigma}(x-y)]$ is only a noise reduction technique. $\Pi^{\rm avg}_{\rho,\sigma}(x-y)$ should remain constant through out the entire calculation.

$\mathbf{M^2}$ trick

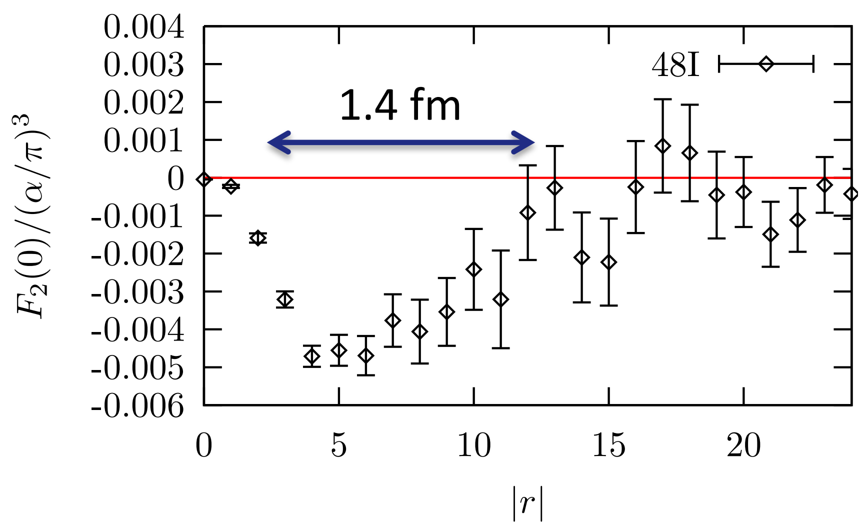


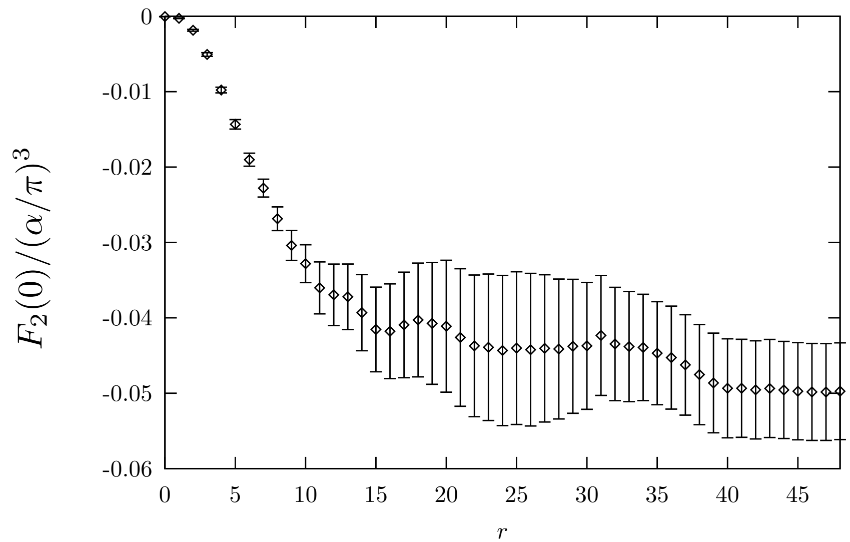
- For QED_L, we can compute the QED function for all z given the y location fixed and x summed over. Allow us to compute all combination of y, z with little efforts.
- For QED_∞, although we can compute all the function $\mathcal{G}_{\rho,\sigma,\kappa}(x, y, z)$ simply by interpolate, we cannot easily compute this function (even after fixing y) for all x and z, simply because of its cost is proportion to Volume^2.
- However, we with QED_∞ and interpolation, we can freely choose which coordinates we compute. For example, we may compute all z for $|x - y| \leqslant 5$, and sample z for $|x - y| > 5$.

140 MeV Pion, disconnected (and connected) LbL results

[Luchang Jin et al. , Phys.Rev.Lett. 118 (2017) 022005]

- left: Integrand function ,   right : Integral





- Using AMA with 2,000 zMobius low modes, AMA

(statistical error only)

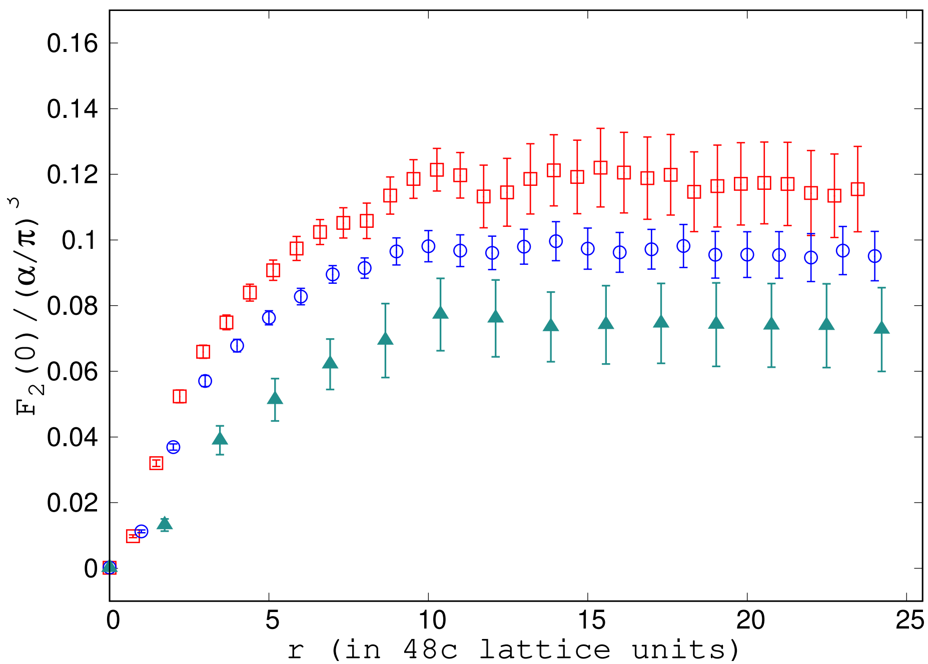
$$\left.\frac{g_\mu - 2}{2}\right|_{\rm cHLbL} = (0.0926 \pm 0.0077) \times \left(\frac{\alpha}{\pi}\right)^3 = (11.60 \pm 0.96) \times 10^{-10}$$

$$\left.\frac{g_\mu - 2}{2}\right|_{\rm dHLbL} = (-0.0498 \pm 0.0064) \times \left(\frac{\alpha}{\pi}\right)^3 = (-6.25 \pm 0.80) \times 10^{-10}$$

$$\left.\frac{g_\mu - 2}{2}\right|_{\rm HLbL} = (0.0427 \pm 0.0108) \times \left(\frac{\alpha}{\pi}\right)^3 = (5.35 \pm 1.35) \times 10^{-10}$$

cHLbL Different lattice spacings

cHLbL: lattice spacing effect (preliminary)



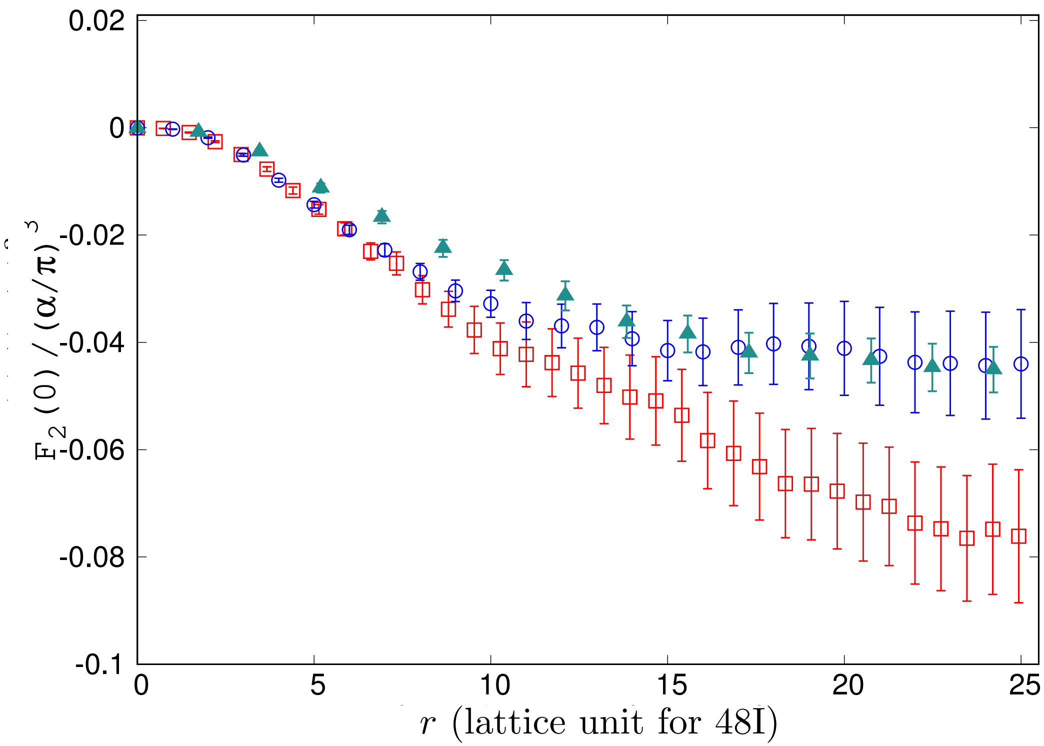
1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV

- Add new 24^3, 1 GeV, ID ensemble (green)
- I and ID slightly different, but disc. errors similar
- Collecting more statistics (9 configs)

- Significant increase as $a \to 0$

dHLbL Different lattice spacings

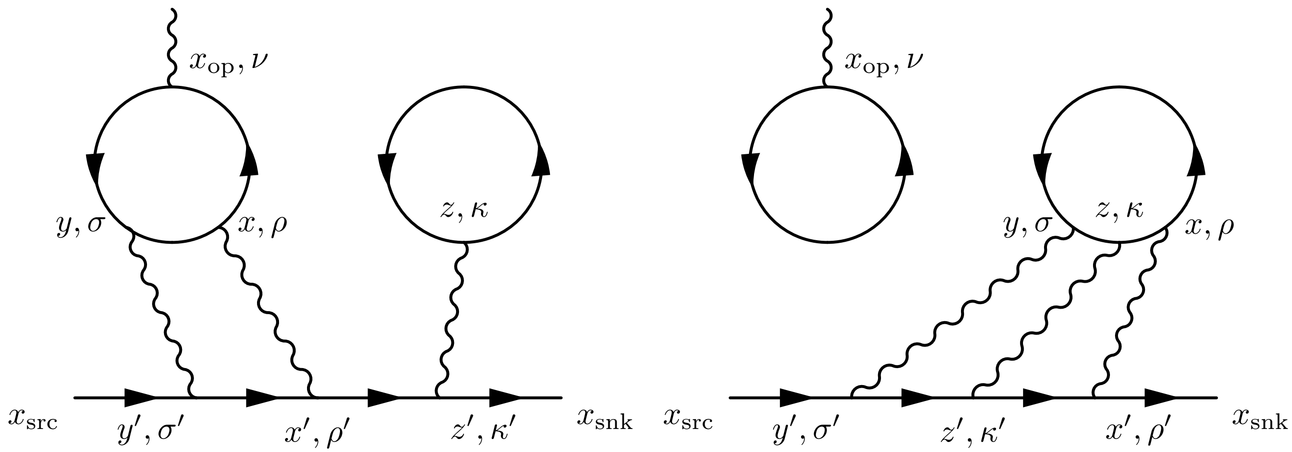
dHLbL contribution: lattice spacing effect (preliminary)



1/a = 2.37 GeV, 1.73 GeV, 1.0 GeV

- Large negative increase tends to cancel connected one
- Collecting more statistics!

Remaining dHLbL



- These are the subleading disconnected diagrams in the SU(3) limit.
- The right diagram has a factor of $1/3$ suppression from the multiplicity of the diagram compare with the left diagram, i.e. the external photon is more likely to be on the loop with three photons.
- For the left diagram, the moment method works just like the connected case. With both QED_L or QED_∞, we can sample x, y and sum over z. We can use the M^2 trick for the x, y sampling. Low-modes-averaging for the loop with z.
- For the right diagram, The moment method still works, however, we have to use a point on the other loop as the reference point, which may be more noisy. But as mentioned above, the right diagram is more suppressed.

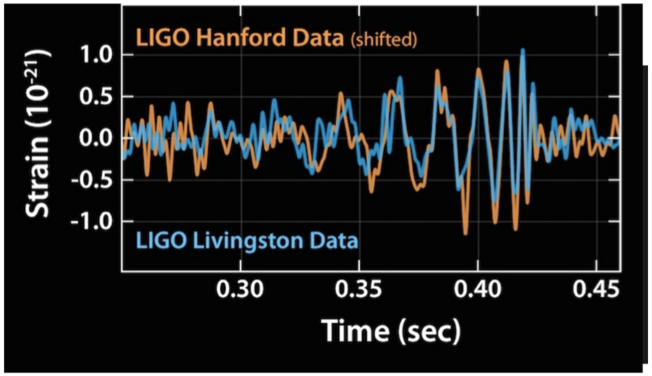
Summary

- Lattice calculation for g-2 calculation is improved very rapidly

- **HLbL** [Luchang Jin et al]
  - computing leading disconnected diagrams :
    -> 8 % stat error in connected, 13 % stat error in leading disconnected
  - coordinate-space integral using analytic photon propagator with adaptive probability (point photon method), config-by-config conserved external current
  - take moment of relative coordinate to directly take $q \to 0$
  - AMA, zMobius, 2000 low modes
  - Infinite volume / continuum QED weight function to avoid power-like FV

- Goal : HLbL 10% error

Can we see the next physics Revolution (c.f GW) ?

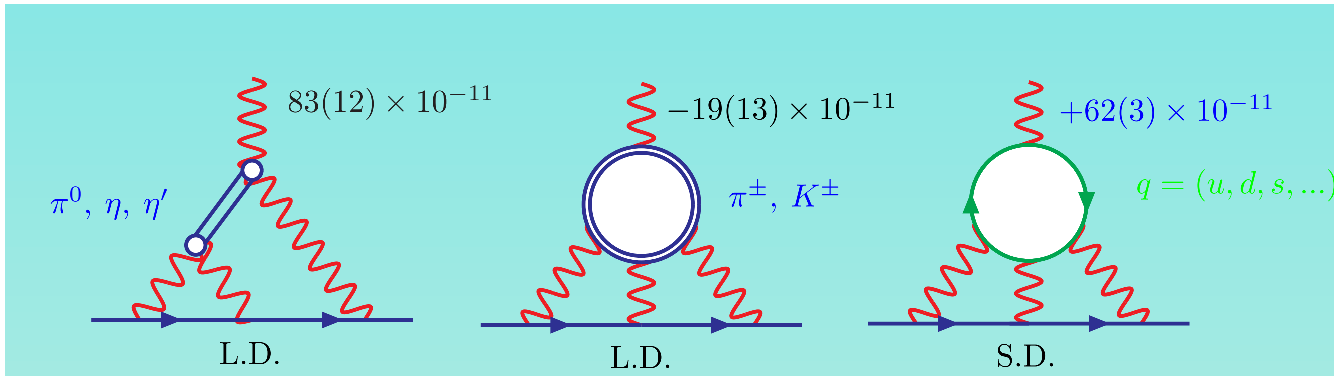


Thank you !

HLbL from Models

- Model estimate with non-perturbative constraints at the chiral / low energy limits using anomaly : (9—12) x 10^{-10} with 25-40% uncertainty

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = 28.8(6.3)_{\rm exp}(4.9)_{\rm SM} \times 10^{-10} \quad [3.6\sigma]$$



F. Jegerlehner , x 10^{11}

Contribution	BPP	HKS	KN	MV	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	0±10	−19±19	−19±13
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	15±10	22± 5
scalars	−6.8±2.0	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	105±26	116±39

Hadronic Light-by-Light (HLbL) contributions