HLbL contribution to the muon g-2 on the lattice: finite volume and discretization effects

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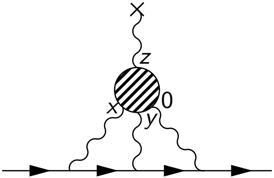
Euclidean position-space approach to a_μ^{HLbL}



master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \Big[\underbrace{\int d^4x}_{=4\pi \int_0^\pi d\beta \sin^2(\beta) \int_0^\infty d|x||x|^3} \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\mathrm{QED}} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\mathrm{QCD}} \Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no finite-volume effects from the photons & affordable way (1d integral) to sample the integrand for the fully connected contribution.
- after contracting all indices the integrand only depends on x^2, $x \cdot y$ and y^2

Finite Volume and Discretization Effects

Choices for $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y)$

- continuum, infinite volume
  $\rightarrow$allows to test the kernel together with an exact correlation function
  $\rightarrow$allows to test discretization and volume effects due to sampling the integrand
  - lepton loop
  - π^0 pole (in the VMD model)
- lattice QCD
  - lattice QED (=lattice QCD without gluonic interactions)
    corresponds to the lepton loop
  - lattice QCD (will be computed in the near future)
    dominated by the π^0 pole

Outline

1. Continuum, Infinite Volume
2. Integration: Finite Lattice $\int \to \sum$, $\quad i\hat{\Pi}$: Continuum, Infinite Volume
3. Full Lattice QED Computation

probe different Kernels (explained later)

Continuum, Infinite Volume

Choices for $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y)$

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  - lattice QCD (will be computed in the near future) dominated by the π^0 pole (model independent)

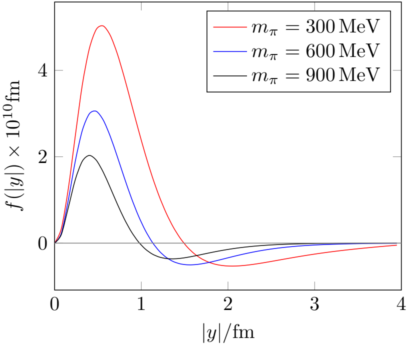
Continuum, Infinite Volume

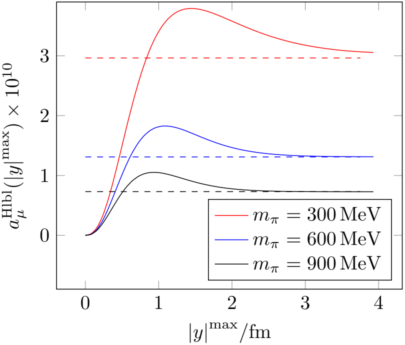
master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 8\pi^3 \int_0^\infty d|y| |y|^3 \Big[\int_0^\infty d|x| |x|^3 \int_0^\pi d\beta \sin^2\beta \, \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \Big].$$

- reduced to 3 dimensional integral
- try different kernels ($\mathcal{L}^{(0,1,2)}$)

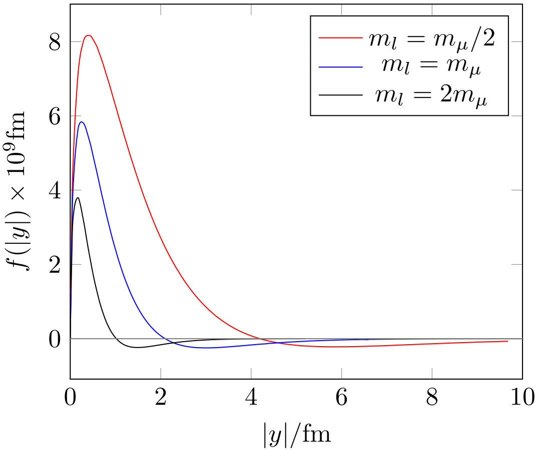
Contribution of the π^0 pole (standard kernel)





- $a_\mu^{HLbL} = \int_0^\infty \mathrm{d}|y| f(|y|)$
- Dashed line = result from momentum-space integration
- Contribution is perhaps surprisingly long-range.

Lepton loop integrand contribution (standard kernel)



- numerically compatible with $f(|y|) \propto m_\mu |y| \log^2(m_\mu |y|)$ for small $|y|$
- analytic result reproduced at the percent level
- peaked at very short distances $\to$ good probe of the kernel there

What next?

achievements

- reproduced π^0-pole contribution
- reproduced lepton loop contribution

challenges in the view of lattice computations

- contributions are quite long range $\rightarrow$ finite volume effects
- integrand peaked at small distances $\rightarrow$ discretization effects

What next?

achievements

- reproduced π^0-pole contribution
- reproduced lepton loop contribution

challenges in the view of lattice computations

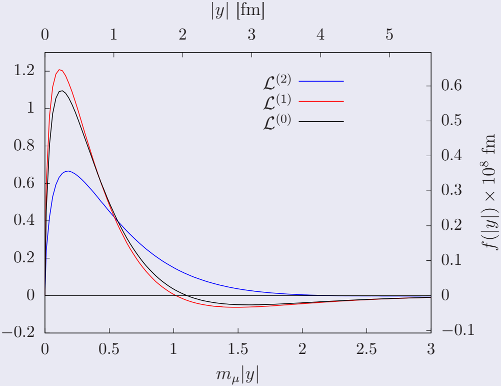
- contributions are quite long range $\rightarrow$ finite volume effects
- integrand peaked at small distances $\rightarrow$ discretization effects

A way to improve

- do subtractions on the kernel (first proposed by Blum *et al* '17)
- we try (short notation):
  - $\mathcal{L}^{(0)} = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ (standard kernel, $\mathcal{L}^{(0)}(0,0) = 0$),
  - $\mathcal{L}^{(1)} = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \frac{1}{2}\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,x) - \frac{1}{2}\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(y,y)$
  - $\mathcal{L}^{(2)} = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(0,y) - \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,0)$

Lepton loop integrand contribution to a_μ^{HLbL}

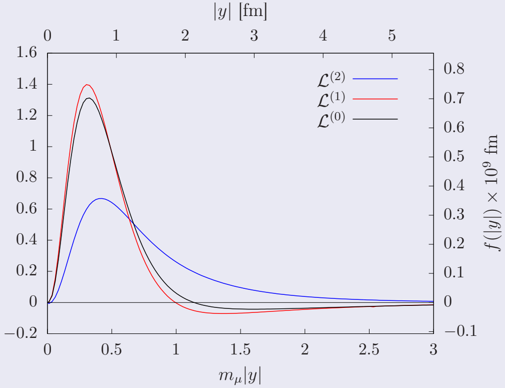
y integrand lepton loop $m_l = m_\mu$



- $\mathcal{L}^{(1)}$ similar to standard kernel, maybe even a bit worse
- $\mathcal{L}^{(2)}$ better: peak less pronounced, falls off faster
  $\rightarrow$ expect reduced lattice artifacts

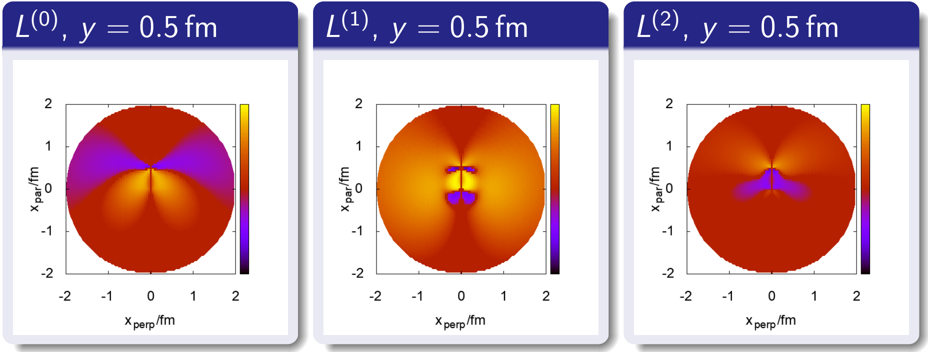
Pion-pole contribution to a_μ^{HLbL}

y integrand pion-pole, $m_\pi = 135\,\mathrm{MeV}$



- very similar observations as for the lepton loop
- again $\mathcal{L}^{(2)}$ seems to be best

Lepton loop integrand contribution to a_μ^{HLbL}



Integration: Finite Lattice $\int \to \sum$, $i\hat{\Pi}$: Continuum, Infinite Volume

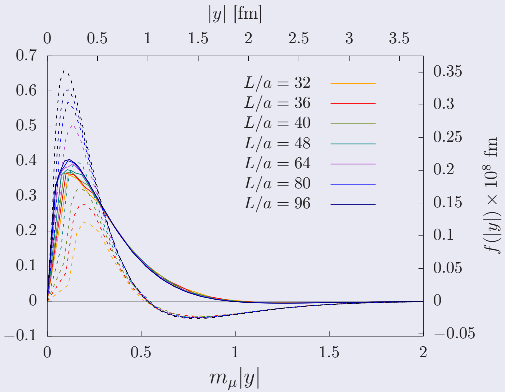
master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 2\pi^2 \sum_{|y|} a_{|y|} |y|^3 \Big[a \sum_{x\in\Lambda} \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \Big].$$

- we can evaluate $\sum_{|y|} a_{|y|}$ on an arbitrary set of $|y|$ and do the integration using e. g. the trapezoidal rule
- the $O(4)$ symmetry allows to choose the direction of y freely
- $i\hat{\Pi}$ still in the continuum
- focus on default kernel $\mathcal{L}^{(0)}$ and kernel $\mathcal{L}^{(2)}$ which seemed to be best

Integration: Finite Lattice $\int \to \sum$, $i\hat{\Pi}$: Continuum, Infinite Volume

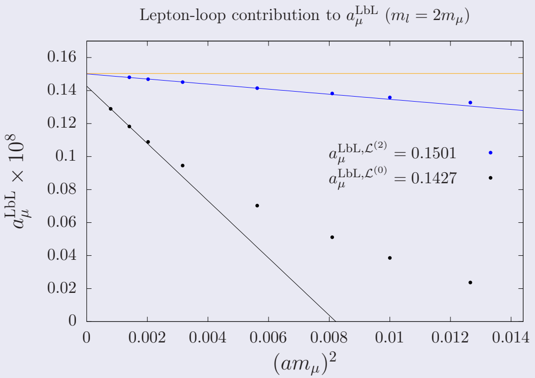
y integrand lepton loop $m_l = 2m_\mu$



- dashed line: default kernel, solid line: $\mathcal{L}^{(2)}$
- constant volume $m_\mu L = 7.2$
- different lattice spacings a

Integration: Finite Lattice $\int \to \sum$, $i\hat{\Pi}$: Continuum, Infinite Volume

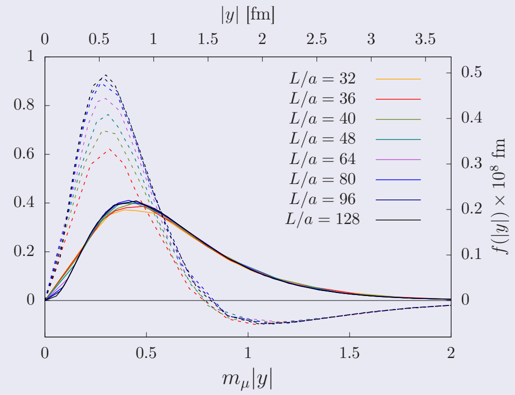
continuum extrapolation lepton loop



- default kernel (black curve): extrapolation is missing some of the curvature
- $\mathcal{L}^{(2)}$ (blue curve): extrapolation works pretty well

Integration: Finite Lattice $\int \to \sum$, $i\hat{\Pi}$: Continuum, Infinite Volume

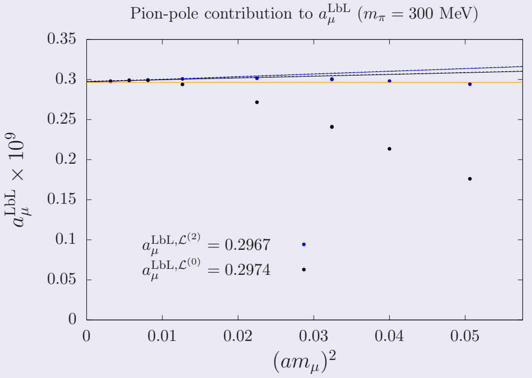
y integrand π^0 pole $m_\pi = 300\,\text{MeV}$



- dashed line: default kernel, solid line: $\mathcal{L}^{(2)}$
- constant volume $m_\mu L = 7.2$
- different lattice spacings a

Integration: Finite Lattice $\int \to \sum$, $i\hat{\Pi}$: Continuum, Infinite Volume

continuum extrapolation π^0 pole



- the discretization effects for the kernel $\mathcal{L}^{(2)}$ are smaller
- but still the extrapolation works very well for both kernels

Full Lattice QED Computation

Choices for $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$

- continuum, infinite volume
  - lepton loop
  - π^0 pole (in the VMD model)
- lattice QCD
  - lattice QED (=lattice QCD without gluonic interactions) corresponds to the lepton loop
  - lattice QCD (will be computed in the near future) dominated by the π^0 pole (model independent)

Full Lattice QED Computation

master formula

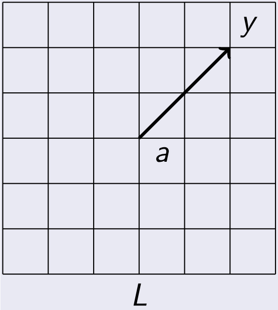
$$a_\mu^{\mathrm{HLbL}} = \frac{me^6}{3} 2\pi^2 \sum_{|y|} a_{|y|} |y|^3 \left[a \sum_{x\in\Lambda} \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \right].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -a \sum_{z\in\Lambda} z_\rho \left\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \right\rangle.$$

- $i\hat{\Pi}$ in lattice QED
- same code as for lattice QCD (just the free case)
- we use local vector currents and conserved vector currents

Full Lattice QED Computation

choice of y

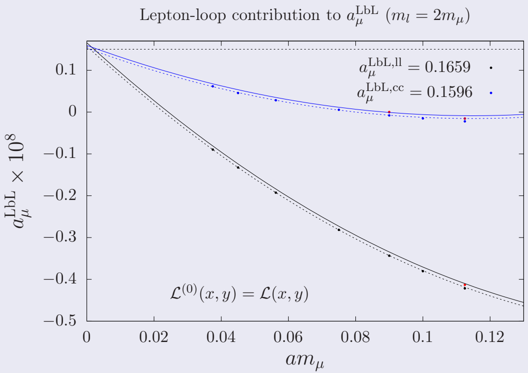


preferred choice $y = (i, i, i, i)$ (lattice diagonal)

- reduced volume effects (distance from border larger than for other choices)
- reduced discretization effects

Full Lattice QED Computation

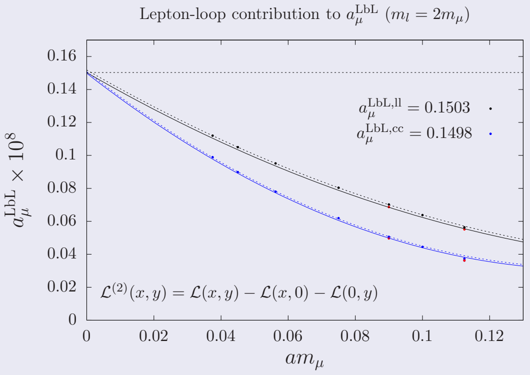
continuum extrapolation lepton loop $\mathcal{L}^{(0)}$



- dashed line: continuum extrapolation for $m_\mu L = 7.2$ using a quadratic fit
- solid line: volume extrapolation: curve shifted by the difference between the results for lattice extents $m_\mu L = 7.2$ and 14.4 at a fixed
- precise extrapolation is difficult for the standard kernel

Full Lattice QED Computation

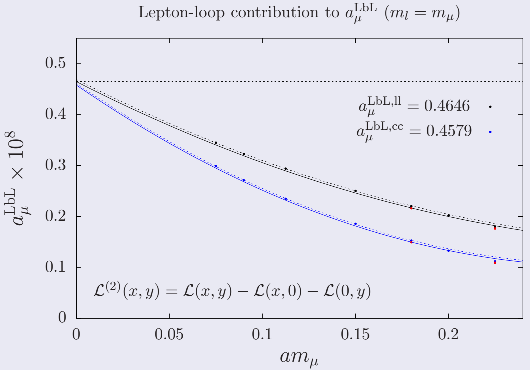
continuum extrapolation lepton loop $\mathcal{L}^{(2)}$



- for the kernel $\mathcal{L}^{(2)}$ the extrapolation works pretty well

Full Lattice QED Computation

continuum extrapolation lepton loop



- the extrapolation works pretty well if all currents are local
- using conserved currents, the extrapolation is a bit low

Conclusion

- Tests of the QED kernel: π^0 pole, lepton loop

- Standard kernel: π^0-pole and lepton-loop contributions are long range and peaked at low distances. This leads to large volume and discretization effects on the Lattice.
- Tried subtractions on the kernel (first proposed by RBC)
  - Analyzed discretization effects in the continuum (π^0 pole, lepton loop)
  - Analyzed discretization effects due to lattice propagators (lepton loop)

  $\rightarrow$ Subtractions help for the lepton loop and the π^0 pole

- We could reproduce the literature result in full lattice QED
- We are ready to start the computation in full lattice QCD

The lepton loop: fully analytic result for $i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$

$$
\begin{aligned}
i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \quad = \quad & \widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) \\
& + \widehat{\Pi}^{(1)}_{\rho;\nu\lambda\mu\sigma}(y-x,-x) + x_\rho\, \Pi^{(r,1)}_{\nu\lambda\mu\sigma}(y-x,-x) \\
& + \widehat{\Pi}^{(1)}_{\rho;\lambda\nu\mu\sigma}(-x,y-x) + x_\rho\, \Pi^{(r,1)}_{\lambda\nu\mu\sigma}(-x,y-x).
\end{aligned}
$$

$$
\begin{aligned}
&\Pi^{(r,1)}_{\mu\nu\lambda\sigma}(x,y) \\
&= 2\Big(\frac{m}{2\pi}\Big)^8 \Big[\frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2} \cdot I_{\gamma\delta}(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|} \cdot p(|y|) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2} \cdot p(|y|) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|} \cdot q_\gamma(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2} \cdot q_\gamma(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+ \frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|} \cdot q_\delta(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2} \cdot q_\delta(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+ \frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|} \cdot I_{\gamma\delta}(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \Big]
\end{aligned}
$$

The lepton loop (continued)

$$
\begin{aligned}
\widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) \quad &= 2\Big(\frac{m}{2\pi}\Big)^8\Big[\frac{(-x_\alpha)(x-y)_\beta K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m|x|)K_1(m|x-y|)}{|x||x-y|}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)(x-y)_\beta\ K_2(m|x|)K_2(m|x-y|)}{|x|^2|x-y|^2}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\, h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\, h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m|x|)K_1(m|x-y|)}{|x|^2|x-y|}\, \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m|x|)K_2(m|x-y|)}{|x||x-y|^2}\, \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big]
\end{aligned}
$$

$$
l_{\gamma\delta}(y) = \frac{2\pi^2}{m^2}\Big(\hat{y}_\gamma\hat{y}_\delta\ K_2(m|y|) - \delta_{\gamma\delta}\ \frac{K_1(m|y|)}{m|y|}\Big), \quad h_{\rho\gamma}(y) = \frac{\pi^2}{m^3}\Big(\hat{y}_\gamma\hat{y}_\rho\ m|y|\ K_1(m|y|) - \delta_{\gamma\rho}K_0(m|y|)\Big),
$$

$$
\hat{f}_{\rho\delta}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\rho\hat{y}_\delta\ m|y|K_1(m|y|) + \delta_{\rho\delta}K_0(m|y|)\Big\} \qquad q_\gamma(y) = \frac{2\pi^2}{m^2}\ \hat{y}_\gamma\ K_1(m|y|),
$$

$$
f_{\rho\delta\gamma}(y) = \frac{\pi^2}{m^3}\Big\{\hat{y}_\gamma\hat{y}_\delta\hat{y}_\rho\ m|y|K_2(m|y|) + (\delta_{\rho\delta}\hat{y}_\gamma - \delta_{\gamma\rho}\hat{y}_\delta - \delta_{\gamma\delta}\hat{y}_\rho)\,K_1(m|y|)\Big\}, \quad p(|y|) = \frac{2\pi^2}{m^2}\ K_0(m|y|).
$$

The π^0 pole contribution

Assume a vector-meson-dominance transition form factor (parameters: m_V, m_π and overall normalization)

$$\mathcal{F}(-q_1^2, -q_2^2) = \frac{c}{(q_1^2 + m_V^2)(q_2^2 + m_V^2)}, \qquad c = -\frac{N_c m_V^4}{12\pi^2 F_\pi}.$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \frac{c^2}{m_V^2(m_V^2 - m_\pi^2)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y_\beta} \Big\{ \epsilon_{\mu\nu\alpha\beta}\epsilon_{\sigma\lambda\rho\gamma} \Big(\frac{\partial}{\partial x_\gamma} + \frac{\partial}{\partial y_\gamma} \Big) K_\pi(x,y)$$
$$+ \epsilon_{\mu\lambda\alpha\beta}\epsilon_{\nu\sigma\gamma\rho} \frac{\partial}{\partial y_\gamma} K_\pi(y-x,y) + \epsilon_{\mu\sigma\alpha\rho}\epsilon_{\nu\lambda\beta\gamma} \frac{\partial}{\partial x_\gamma} K_\pi(x, x-y) \Big\}.$$

where

$$K_\pi(x,y) \equiv \int d^4u \Big(G_{m_\pi}(u) - G_{m_V}(u) \Big) G_{m_V}(x-u) G_{m_V}(y-u) = K_\pi(y,x).$$