Determining the long-distance contribution to the HLbL portion of *g*-2 in position space from the π^0 pole

Muon g-2 Theory Initiative Hadronic
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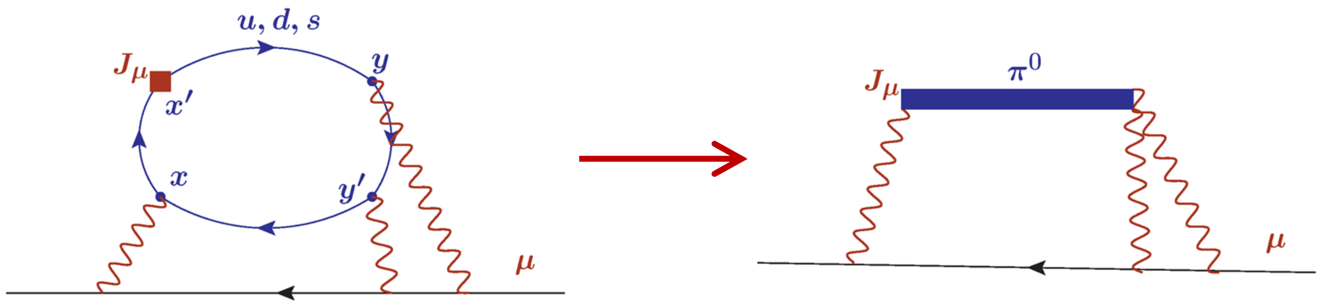
Overview

- Long distance contribution to HLbL comes from the π^0 exchange.

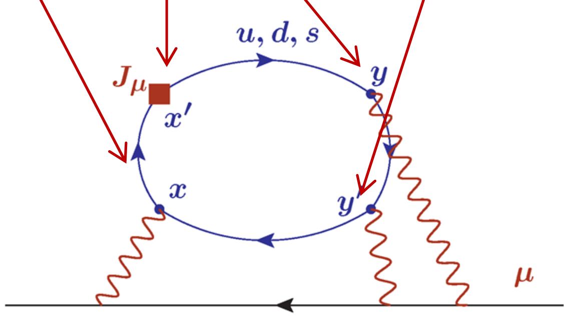


- Calculate π^0 exchange from lattice QCD
  - Direct calculation, without form factor decomposition or parameterization.
  - Position-space based.
  - Can be applied for large volume.
- No implementation at present.

Introduce π^0 states

- Compute the π^0 pole contribution to:

$$\mathcal{A}_{\mu\mu'\nu\nu'}(x,x',y,y') = \langle 0 | T\big(J_\mu(x) J_{\mu'}(x') J_\nu(y) J_{\nu'}(y')\big) | 0 \rangle$$



- Assume x and y are far separated in the time direction and insert sum over π^0 states:

$$\mathcal{A}^{\pi^0}_{\mu_1\mu_2\nu_1\nu_2}(x,x',y,y') = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_\pi(p)} \langle 0 | T\big(J_\mu(x) J_{\mu'}(x')\big) | \pi^0(\vec{p}) \rangle \langle \pi^0(\vec{p}) | T\big(J_\nu(y) J_{\nu'}(y')\big) | 0 \rangle$$

- Dominant contribution for x_0-y_0 large.

Use translational symmetry

$$\mathcal{A}^{\pi^0}_{\mu_1\mu_2\nu_1\nu_2}(x,x',y,y') \;=\; \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_\pi(p)}\langle 0\big|T\big(J_\mu(x)J_{\mu'}(x')\big)\big|\pi^0(\vec{p})\rangle \langle \pi^0(\vec{p})\big|T\big(J_\nu(y)J_{\nu'}(y')\big)\big|0\rangle$$

$$\begin{aligned}\langle 0\big|T\big(J_\mu(x)J_{\mu'}(x')\big)\big|\pi^0(\vec{p})\rangle \quad &= \langle 0\big|T\big(J_\mu(0)J_{\mu'}(\widetilde{x})\big)\big|\pi^0(\vec{p})\rangle e^{i\vec{p}\cdot\vec{x}-E_p x_0} \\ &= \mathcal{F}_{\mu\mu'}(\widetilde{x},\vec{p})e^{i\vec{p}\cdot\vec{x}-E_p x_0} \\ &= \mathcal{F}_{\mu\mu'}\left(\widetilde{x},-i\vec{\nabla}_x\right) e^{i\vec{p}\cdot\vec{x}-E_p x_0}\end{aligned}$$

$$E_p = \sqrt{\vec{p}^{\,2}+M_\pi^2}$$

- After these standard steps the integral over $\vec{p}$ can be performed, giving the Euclidean pion propagator:

$$\begin{aligned}\frac{1}{(2\pi)^3}\int \frac{d^3p}{2E_\pi(p)}e^{i\vec{p}\cdot(\vec{x}-\vec{y})-E_p(x_0-y_0)} \;&=\; \frac{1}{(2\pi)^4}\int d^4p\frac{e^{ip(x-y)}}{p_0^2+\vec{p}^{\,2}+M_\pi^2} \\ &\equiv\; \Delta_F(x-y,M_\pi)\end{aligned}$$

Further long-distance approximation

- Combine these results to obtain:

$$\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x,x',y,y') = \mathcal{F}_{\mu\mu'}\left(\widetilde{x}, -i\vec{\nabla}_x\right)\mathcal{F}_{\nu\nu'}\left(\widetilde{y}, i\vec{\nabla}_y\right)\Delta_F(x-y, M_\pi)$$

- Next evaluated the spatial derivatives:

$$\begin{aligned}\prod_{i=1}^{N}\left(\frac{\partial}{\partial x_{\rho_i}}\right)\Delta_F(x-y,M_\pi) &= \left\{\prod_{i=1}^{N}\left(\frac{\partial}{\partial x_{\rho_i}}\right)\right\}\left(\frac{M_\pi}{2\pi|x-y|}\right)^{3/2} e^{-|x-y|M_\pi} \\ &\approx \left\{\prod_{i=1}^{N}\left(-M_\pi\frac{(x-y)_{\rho_i}}{|x-y|}\right)\right\}\left(\frac{M_\pi}{2\pi|x-y|}\right)^{3/2} e^{-|x-y|M_\pi}\end{aligned}$$

- Which implies:

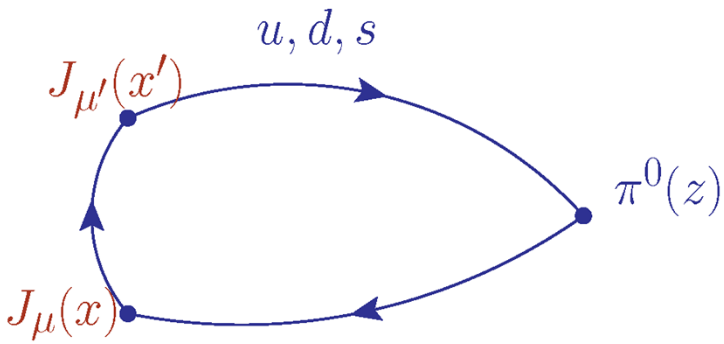
$$\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x,x',y,y') = \mathcal{F}_{\mu\mu'}\left(\widetilde{x}, iM_\pi\hat{n}\right)\mathcal{F}_{\nu\nu}\left(\widetilde{y}, -iM_\pi\hat{n}\right)\Delta_F(x-y, M_\pi)$$

where $\hat{n} = \dfrac{\vec{x}-\vec{y}}{|x-y|}$, a unit Euclidean four-vector.

Calculate $\gamma\gamma$ - pion vertex directly

- The amplitude $\mathcal{F}_{\mu\mu'}(\tilde{x}, iM_\pi \hat{n})$ also appears in a simpler Green's function:

$$\mathcal{B}_{\mu\mu'}(x, x', z) = \langle 0 | T\big(J_\mu(x) J_{\mu'}(x') \pi^0(z)\big) | 0 \rangle$$



- This can be directly evaluated using lattice QCD

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$$\mathcal{B}_{\mu\mu'}(x, x', z) = \langle 0 | T\big(J_\mu(x) J_{\mu'}(x') \pi^0(z)\big) | 0 \rangle$$

- Involves the same $\gamma\gamma - \pi$ vertex as $\mathcal{A}_{\mu\mu'\nu\nu'}(x, x', y, y')$

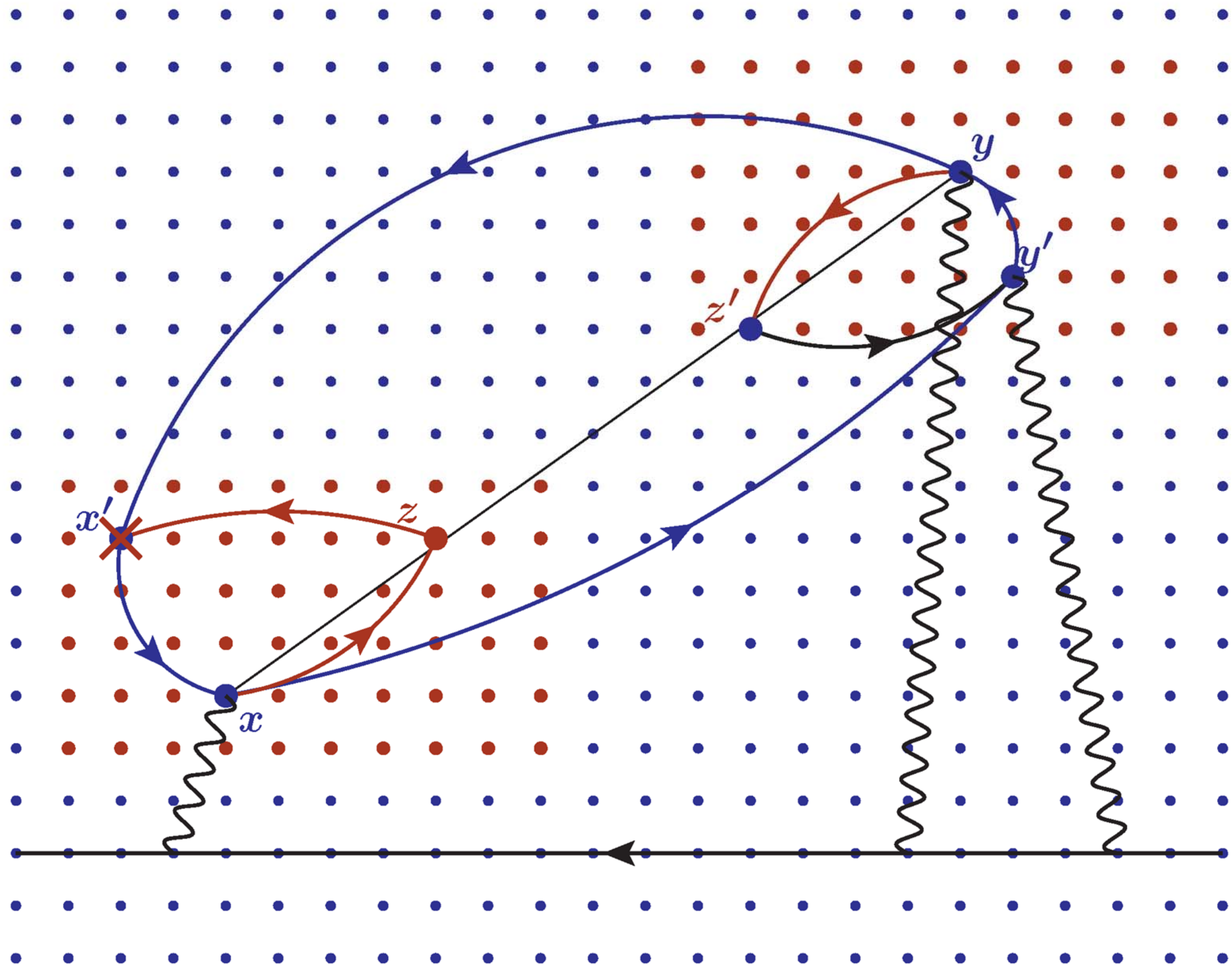
$$\mathcal{B}^{\pi^0}_{\mu\mu'}(x, x', z) = \mathcal{F}_{\mu\mu'}(\widetilde{x}, iM_\pi\hat{n})\, Z^{1/2}_{\pi^0} \Delta_F(x - z, M_\pi)$$

where $Z^{1/2}_{\pi^0} = \langle \pi^0(\vec{p} = 0) | \pi^0(0) | 0 \rangle$

- Combining results for $\mathcal{A}^{\pi^0}_{\mu_1\mu_2\nu_1\nu_2}(x, x', y, y')$ and $\mathcal{B}^{\pi^0}_{\mu\mu'}(x, x', z)$

$$\mathcal{A}^{\pi^0}_{\mu\mu'\nu\nu'}(x, x', y, y') = \mathcal{B}^{\pi^0}_{\mu\mu'}(x, x', z) \mathcal{B}^{\pi^0}_{\nu\nu'}(y, y', z') \frac{1}{Z_{\pi^0}} \frac{\Delta_F(x - y, M_\pi)}{\Delta_F(x - z, M_\pi) \Delta_F(z' - y, M_\pi)}$$

Lattice implementation



π^0 contribution to $(g\text{-}2)_{\text{HLbL}}$

- Combine photon and muon propagators $\mathcal{G}_{\rho,\sigma,\kappa}(x,y,y')$ with the QCD part to obtain:

$$\frac{1}{2m_\mu}F_2^{\pi^0}(q^2=0)(\sigma_{s',s})_i$$
$$=\frac{1}{VT}\sum_{x,x',y,y'}\frac{(-ie)^6}{2}\epsilon_{i,j,k}\left(x'-\frac{x+y}{2}\right)_j$$
$$\cdot i\overline{u}_{s'}(\vec{0})\mathcal{G}_{\rho,\sigma,\kappa}(x,y,y')u_s(\vec{0})\mathcal{A}^{\pi^0}(x,x',y,y')_{\rho,\sigma,\kappa,k}$$

- For $|x\text{-}y| \geq R_{\text{min}}$ replace $\mathcal{A}_{\mu\mu'\nu\nu'}(x,x',y,y')$ with $\mathcal{A}^{\pi^0}_{\mu_1\mu_2\nu_1\nu_2}(x,x',y,y')$ to use the π^0 contribution at long distances.

Conclusion

- Contribution of π^0 exchange to HLbL at long distances:
  - Is well-defined in an *x*-space calculation.
  - Can be precisely computed from lattice QCD.
  - A fixed volume QCD calculation gives the π^0 HLbL contribution in increasing volume.
- Should allow the large volume systematic error to be reduced.