HADRONIC CONTRIBUTIONS
VIA
THE SCHWINGER SUM RULE

Vladimir Pascalutsa

Institute for Nuclear Physics
University of Mainz, Germany

with
Franziska Hagelstein

Albert Einstein Centre
University of Bern, Germany

THE LOW-ENERGY FRONTIER
OF THE STANDARD MODEL

@ HLbL Workshop
UConn, Storrs
March 12-14, 2018

PHYSICAL REVIEW LETTERS **120**, 072002 (2018)

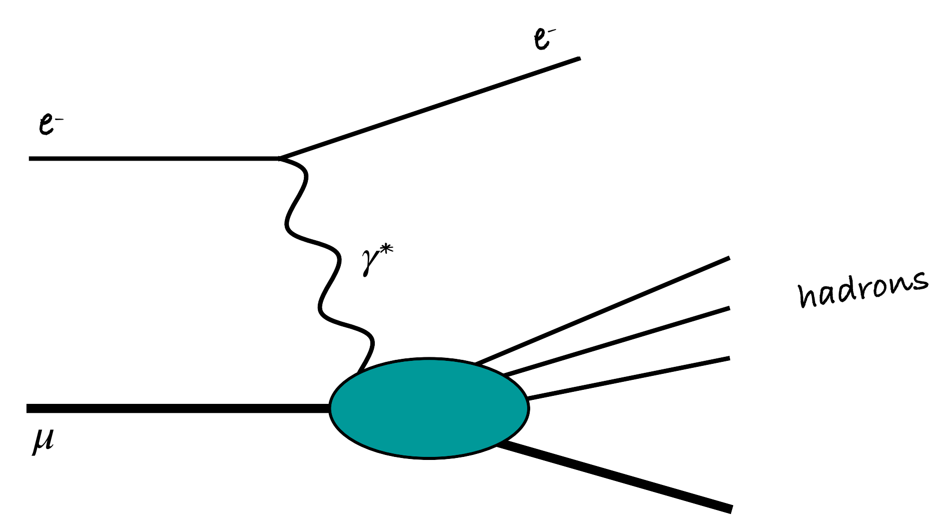
Dissecting the Hadronic Contributions to $(g-2)_\mu$ by Schwinger's Sum Rule

Franziska Hagelstein[1,2] and Vladimir Pascalutsa[1]
[1]*Institut für Kernphysik & Cluster of Excellence PRISMA, Johannes Gutenberg-Universität Mainz, D-55128 Mainz, Germany*
[2]*Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, CH-3012 Bern, Switzerland*

Outline

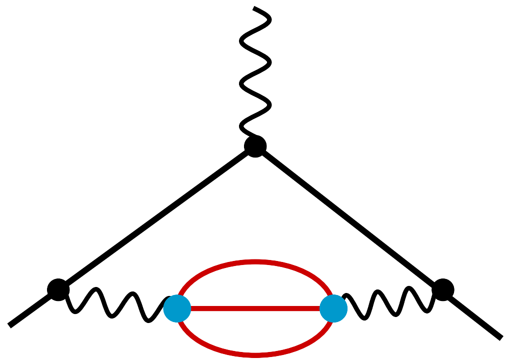
$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$
$$= \lim_{Q^2\to 0}\frac{8m^2}{Q^2}\int_0^{x_0}\mathrm{d}x\,[\bar{g}_1+\bar{g}_2](x,Q^2)$$

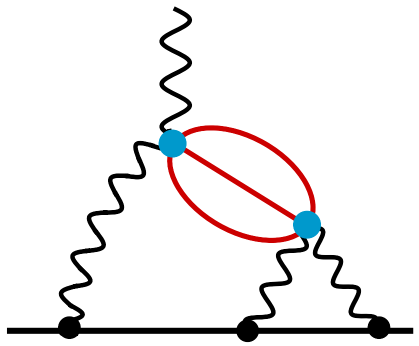
- THE SCHWINGER SUM RULE
- Reproducing $\varkappa^{(1)} = \alpha/2\pi$
- HADRONIC *VACUUM POLARIZATION* AND *LIGHT-BY-LIGHT* CONTRIBUTIONS ON THE SAME FOOTING
- *MUON STRUCTURE FUNCTIONS* FROM INELASTIC MUON-ELECTRON SCATTERING



Motivation

- Uncertainty of the SM prediction for the muon anomaly $(g-2)_\mu$ is dominated by hadronic contributions (HVP and HLbL)





- HVP is calculated with a data-driven dispersive approach:

$$\varkappa^{\mathrm{HVP}} = \frac{\alpha}{\pi^2} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}s}{s} \, \mathrm{Im}\, \Pi^{\mathrm{had}}(s)\, K(s/m^2)$$

$$\mathrm{Im}\, \Pi^{\mathrm{had}}(s) = \frac{s}{4\pi\alpha} \sigma(\gamma^* \to \text{anything})$$

F. Jegerlehner, Springer Tracts Mod. Phys. 274 (2017).

M. Davier, Nucl. Part. Phys. Proc. 287-288, 70 (2017)

- HLbL is not as simple, data-driven, systematic

- Schwinger's sum rule is an exact dispersive formula which treats HVP and HLbL (and everything else) in **the same way.**

Schwinger Sum Rule

J. S. Schwinger, Proc. Nat. Acad. Sci. 72, 1 (1975); ibid. 72, 1559 (1975) [Acta Phys. Austriaca Suppl. 14, 471 (1975)].
A. M. Harun ar-Rashid, Nuovo Cim. A 33, 447 (1976).

anomalous magnetic moment (a.m.m.) $\varkappa = 1/2\,(g-2)_\mu$

muon mass m

photon lab-frame energy ν and virtuality $Q^2 = -q^2$

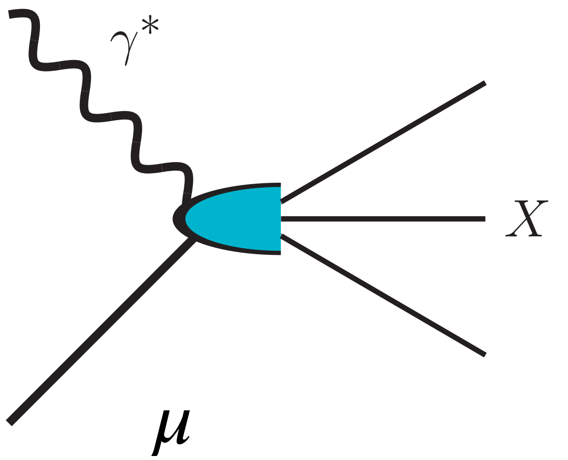
$$\varkappa = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} d\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2=0}$$

fine-structure constant $\alpha \approx 1/137$

photo-absorption threshold ν_0

longitudinal-transverse photo-absorption cross section σ_{LT}

- photo-absorption on muon:



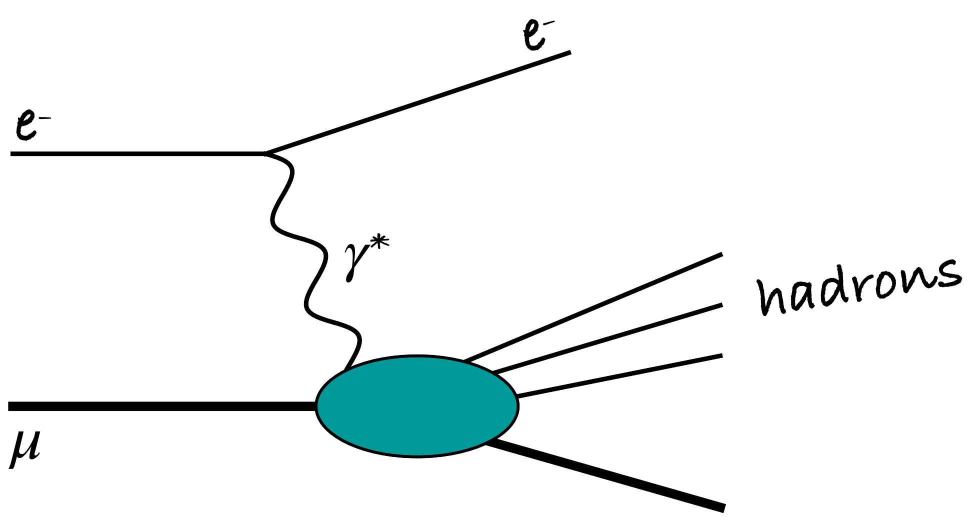
$= \gamma\mu,\ \gamma\gamma\mu,\ \pi^0\mu,\ \gamma\pi^0\mu,\ \ldots$

Spin structure functions

a.m.m.
$\varkappa = 1/2 (g-2)_\mu$

$$\varkappa = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} d\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2=0}$$

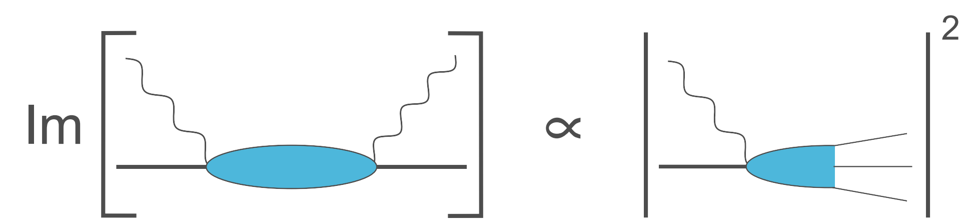
$$= \lim_{Q^2 \to 0} \frac{8m^2}{Q^2} \int_0^{x_0} dx\, [\bar{g}_1 + \bar{g}_2](x, Q^2)$$



muon spin structure functions
g_1 and g_2

- Spin-dependent forward doubly-virtual Compton scattering:

$$T_A^{\mu\nu}(q,p) = -\frac{1}{M} \gamma^{\mu\nu\alpha} q_\alpha\, S_1(\nu, Q^2) + \frac{Q^2}{M^2} \gamma^{\mu\nu} S_2(\nu, Q^2)$$



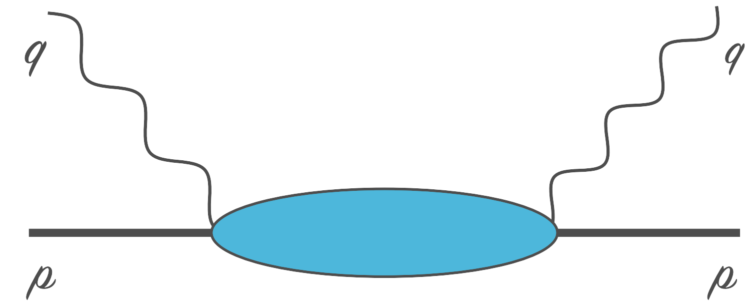
- Optical theorem:

$$\mathrm{Im}\, S_1(\nu, Q^2) = \frac{4\pi^2 \alpha}{\nu} g_1(x, Q^2) = \frac{M\nu^2}{\nu^2 + Q^2} \left[\frac{Q}{\nu} \sigma_{LT} + \sigma_{TT} \right] (\nu, Q^2)$$

$$\mathrm{Im}\, S_2(\nu, Q^2) = \frac{4\pi^2 \alpha M}{\nu^2} g_2(x, Q^2) = \frac{M^2\nu}{\nu^2 + Q^2} \left[\frac{\nu}{Q} \sigma_{LT} - \sigma_{TT} \right] (\nu, Q^2)$$

Origin

- Sum rules are model-independent relations based on general principles of:
  - Analyticity/causality (dispersion relations),
  - unitarity (optical theorem)
  - crossing symmetry



- Examples of sum rules include:

$$(1+\varkappa)\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}}{Q} - \frac{\sigma_{TT}}{\nu}\right]_{Q^2=0}$$

Burkhardt—Cottingham sum rule (1970) $\int_0^1 \mathrm{d}x\, g_2(x,Q^2) = 0$

$\ominus$

$$\varkappa^2 = -\frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu\, \frac{\sigma_{TT}(\nu)}{\nu}$$

Gerasimov—Drell—Hearn sum rule (1966)

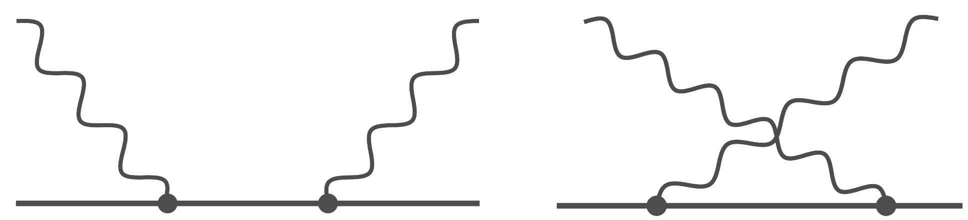
$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

Schwinger sum rule (1975)

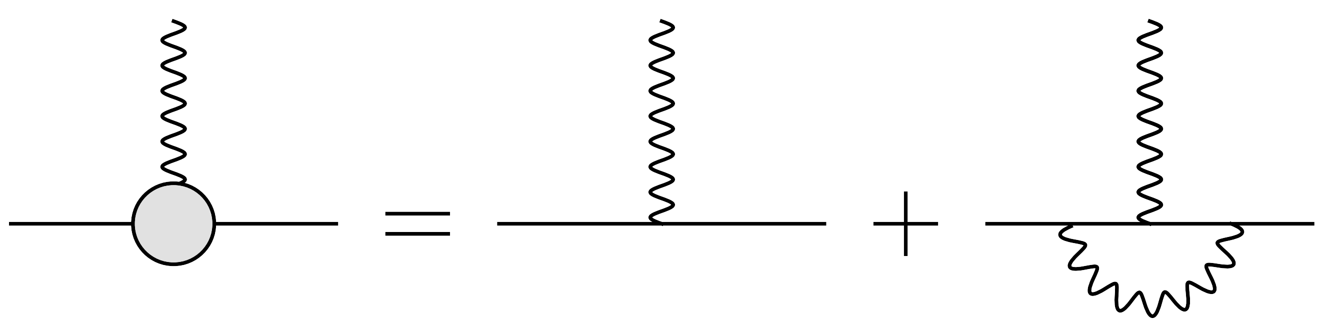
Reproducing the leading QED result

- Schwinger sum rule: $\varkappa = \frac{m^2}{\pi^2 \alpha} \int_{\nu_0}^{\infty} \mathrm{d}\nu \left[\frac{\sigma_{LT}(\nu, Q^2)}{Q} \right]_{Q^2=0}$
- Input: longitudinal-transverse photo-absorption cross section

tree-level QED
Compton scattering



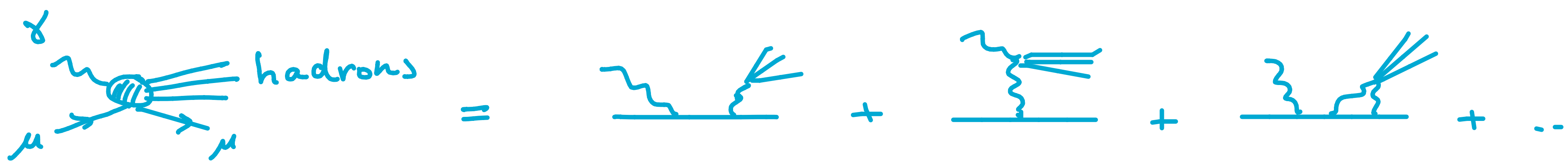
$$\sigma_{LT}^{\gamma^* \mu \to \gamma \mu}(\nu, Q^2) = \frac{\pi \alpha^2 Q\,(s - m^2)^2}{4 m^3 \nu^2 \left(\nu^2 + Q^2\right)} \left(-2 - \frac{m(m+\nu)}{s} + \frac{3m + 2\nu}{\sqrt{\nu^2 + Q^2}} \,\mathrm{arccoth} \frac{m+\nu}{\sqrt{\nu^2 + Q^2}} \right)$$



$F_2(0) = \varkappa$ $\quad$ $\varkappa^{(0)} = 0$ $\quad$ $\varkappa^{(1)} = \alpha / 2\pi$

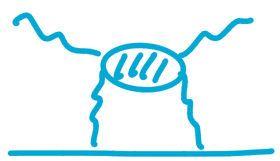
Hadronic Contributions:
4 channels to order α^3

$\gamma\mu \to \mu + \text{hadrons}$

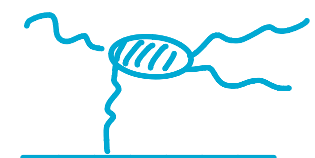


$\gamma\mu \to \gamma\mu + \text{hadrons}$

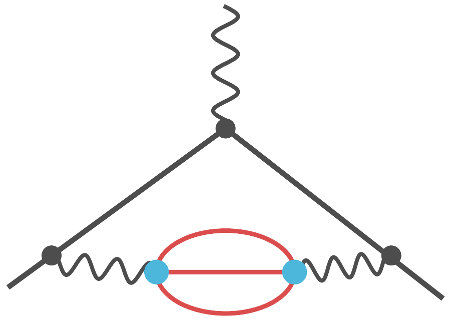
$\gamma\mu \to \gamma\mu$

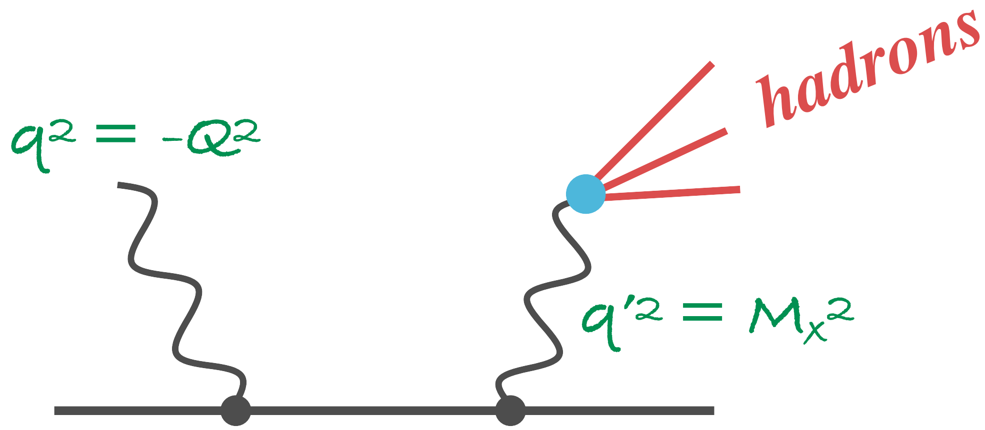


$\gamma\mu \to \gamma\gamma\mu$



Reproducing the HVP formula





+ crossed diagram

- Cross section of hadron production through timelike Compton scattering:

factorizes as:

$$\sigma(\gamma\mu \to \mu X) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}M_X^2}{M_X^2}\,\sigma(\gamma\mu \to \gamma^*\mu)\,\mathrm{Im}\,\Pi_X(M_X^2)$$

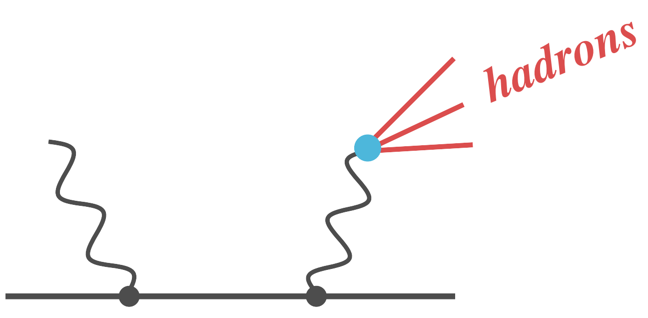
timelike Compton scattering — virtual-photon decay into hadrons

- Timelike Compton scattering cross section:

$$\left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu,Q^2)}{Q}\right]_{Q^2=0} = \frac{\pi\alpha^2}{2m^2\nu^3}\left[-(5s+m^2+M_X^2)\lambda + (s+2m^2-2M_X^2)\log\frac{\beta+\lambda}{\beta-\lambda}\right]$$

$$\beta = (s+m^2-M_X^2)/2s \qquad s = m^2+2m\nu$$
$$\lambda = (1/2s)\sqrt{[s-(m+M_X)^2]\,[s-(m-M_X)^2]}$$

HVP from Schwinger sum rule



$$\varkappa = \frac{m^2}{\pi^2\alpha}\int_{4m_\pi^2}^{\infty} dM_X^2 \int_{\nu_0}^{\infty} d\nu \left[\frac{1}{Q}\frac{d\sigma_{LT}^{\gamma\mu\to\mu X}(\nu,Q^2)}{dM_X^2}\right]_{Q^2=0}$$

$$= \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} dM_X^2 \,\frac{\mathrm{Im}\,\Pi^{\mathrm{had}}(M_X^2)}{M_X^2}\,\frac{m^2}{\pi^2\alpha}\int_{\nu_0}^{\infty} d\nu \left[\frac{\sigma_{LT}^{\gamma\mu\to\gamma^*\mu}(\nu,Q^2)}{Q}\right]_{Q^2=0}$$

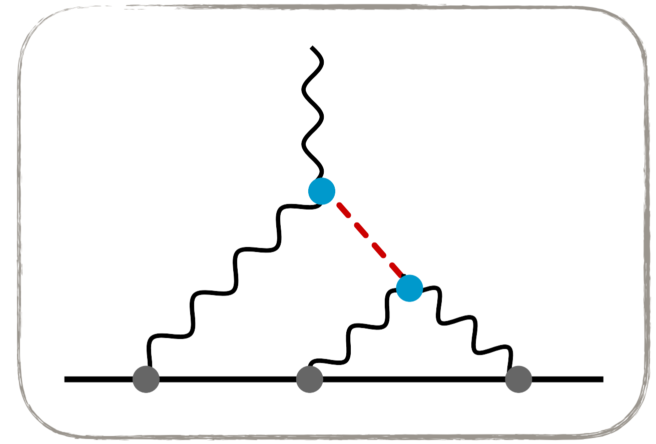
kernel function: $= \frac{\alpha}{\pi}K(M_X^2/m^2) \equiv \frac{\alpha}{\pi}\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)(M_X^2/m^2)}$

for $M_x=0$, we find $K(0)=1/2$, and therefore
the Schwinger term: $\varkappa^{(1)} = \alpha/2\pi$

- reproduces the HVP standard formula

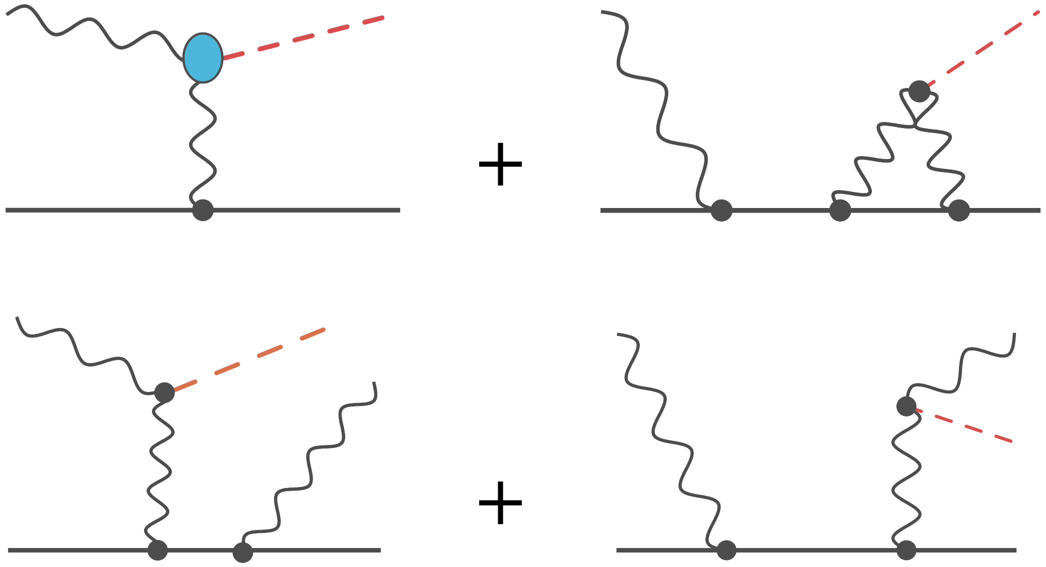
$$\varkappa^{\mathrm{HVP}} = \frac{\alpha}{\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}\,\mathrm{Im}\,\Pi^{\mathrm{had}}(s)\int_0^1 dx\,\frac{x^2(1-x)}{x^2+(1-x)(s/m^2)}$$

Light-by-Light contributions

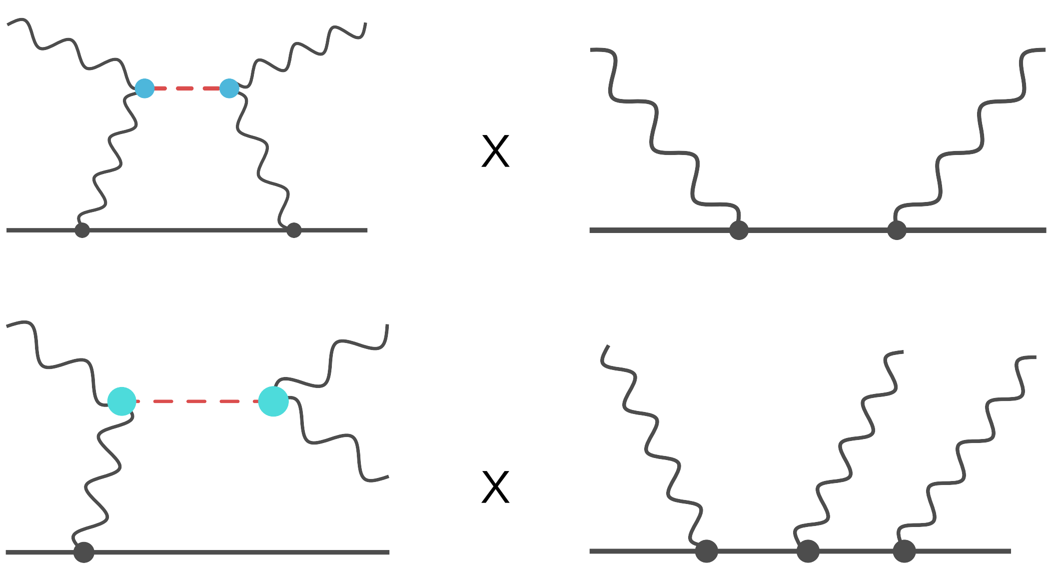


(pseudo-)scalar contribution

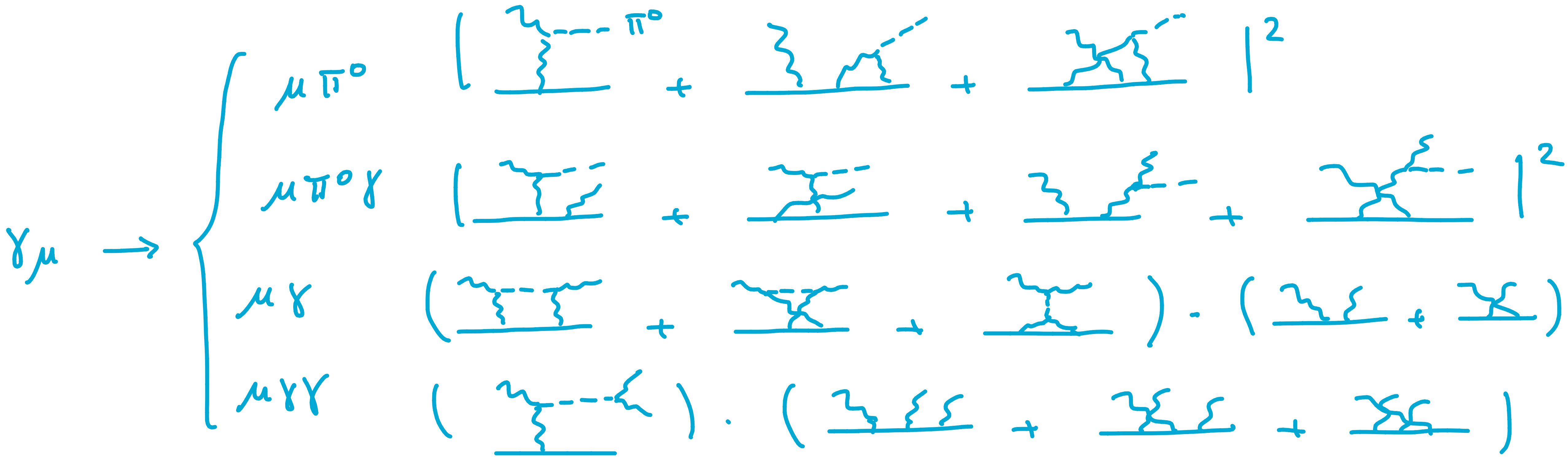
I. Hadron photo-production channels



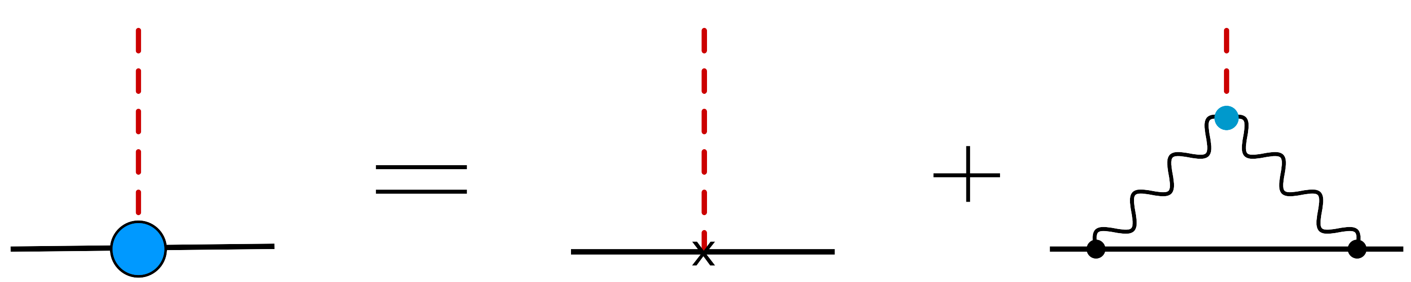
II. Electromagnetic channels



Paseudo-scalar contribution in full glory



- **No doubly-virtual transition form factors needed, if hadronic channels are measured**

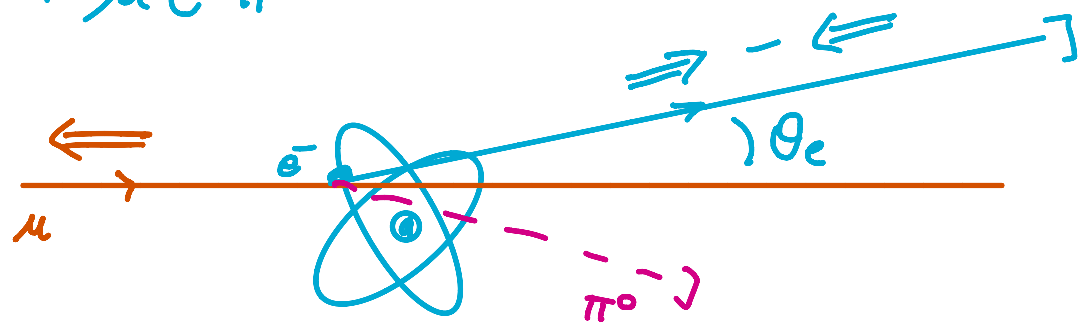


Pseudoscalar meson coupling to leptons.

Feasibility of measurement at COMPASS as part of MUonE ?

cf. The Workshop on
Evaluation of the Leading Hadronic Contribution
to the Muon Anomalous Magnetic Moment
Mainz (Germany), 2 - 5 April 2017

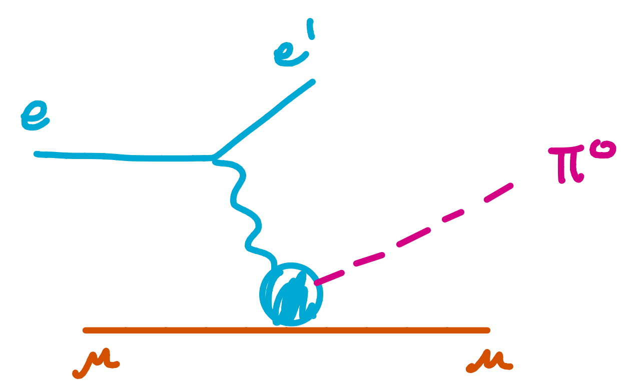
$\mu e \to \mu e \pi^0$



$E_\mu = 150, \; 200 \text{ GeV}$

$E'_e \simeq 1 \text{ GeV}$

$\theta_e \simeq 10 \text{ mrad}$

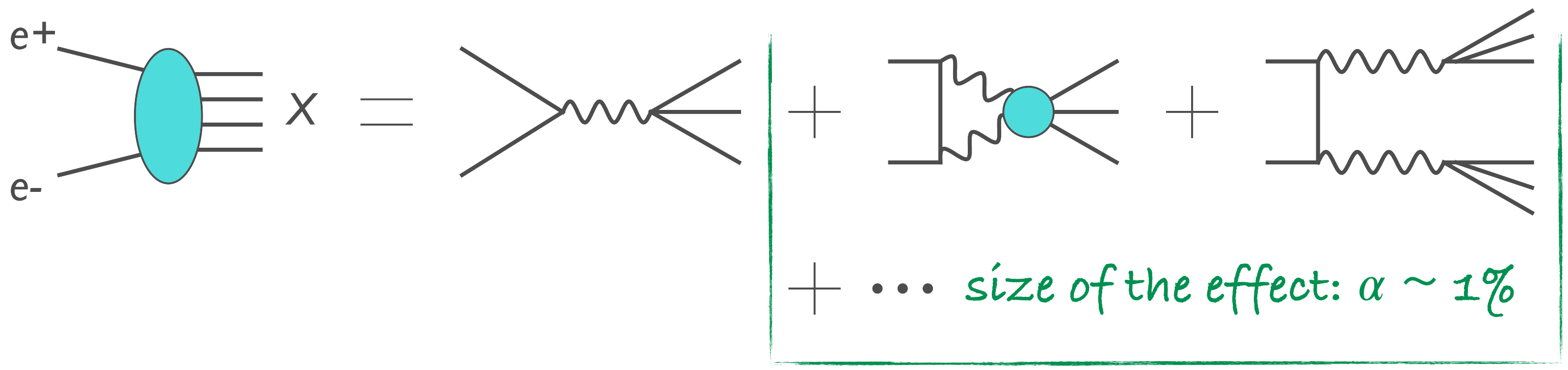


$$Q^2 \simeq 2 m_e E'_e \simeq 10^{-3} \text{ GeV}^2$$

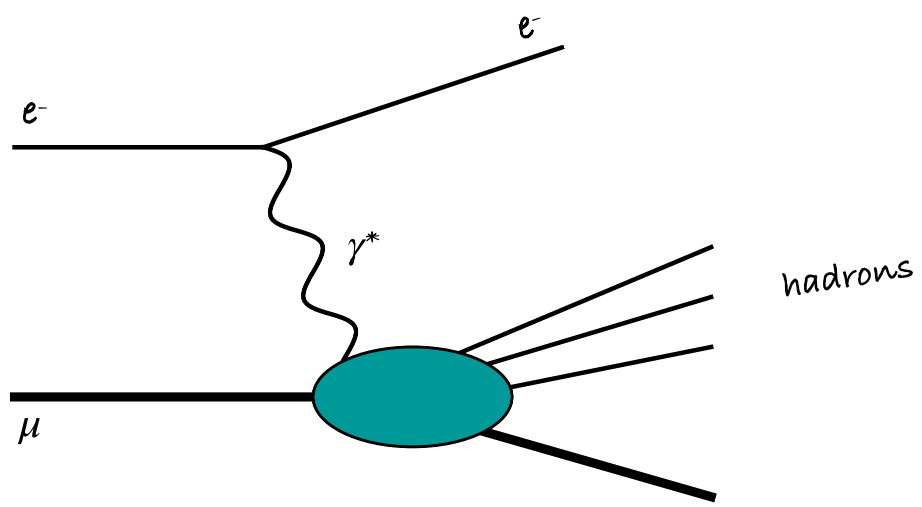
$$\nu \simeq \frac{m_e E_\mu}{m_\mu} \left(1 - 2 \frac{E'_e}{m_e} \sin^2 \frac{\theta}{2} \right) = \left(\nu_{\pi^0}, \; 1 \text{ GeV} \right)$$

$$\nu_{\pi^0} = \frac{m_{\pi^0}}{m_\mu} \left(\tfrac{1}{2} m_{\pi^0} + m_\mu \right) \simeq 230 \text{ MeV}$$

Possible refinements of the HVP



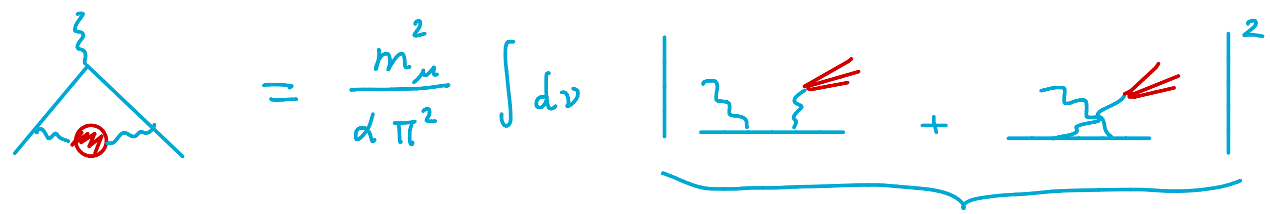
vs.



Summary and Conclusions

1. Schwinger sum rule — dispersive formula applying equally to HVP and HLbL

2. Reproduces $\alpha/2\pi$ and HVP formula:



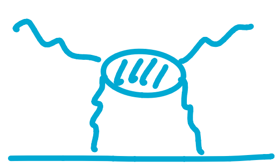
$$a_\mu^{HVP} = \frac{m_\mu^2}{\alpha\,\pi^2}\int d\nu \int dM_X^2\; \sigma_{LT}(\gamma\mu \to \gamma_X^*\mu)\;\Gamma(\gamma_X^* \to \text{hadrons})$$

3. Splits contributions into hadron production and e.m. (LbL) channels

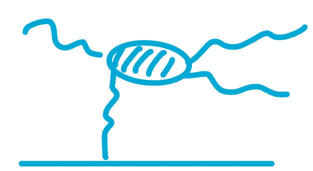
$\gamma\mu \to \mu + \text{hadrons}$

measurable

$\gamma\mu \to \gamma\mu$



$\gamma\mu \to \gamma\gamma\mu$



calculable by lattice

4. Near future:
evaluate the PS contribution by plenary meeting in Mainz

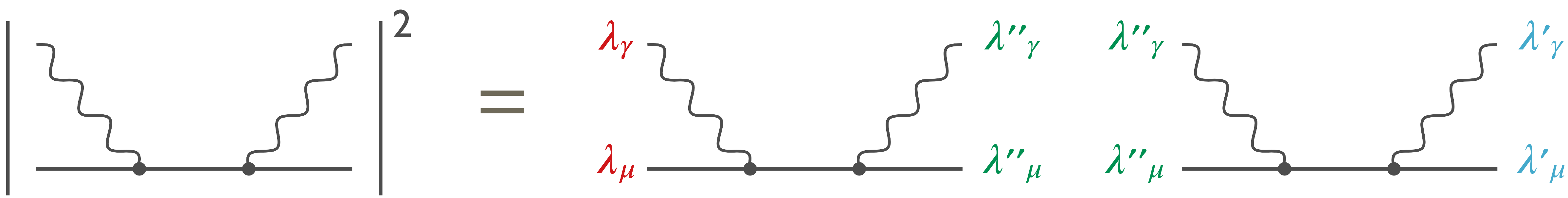
Backup slides

The Cross section σ_{LT}

- Example: tree-level QED Compton scattering cross section

$$d\sigma_{\lambda'_\gamma \lambda'_\mu \lambda_\gamma \lambda_\mu} = (2\pi)^4 \delta^{(4)}(p_f - p_i) \sum_{\lambda''_\gamma, \lambda''_\mu} \frac{\mathcal{M}^\dagger_{\lambda'_\gamma \lambda'_\mu \lambda''_\gamma \lambda''_\mu} \mathcal{M}_{\lambda''_\gamma \lambda''_\mu \lambda_\gamma \lambda_\mu}}{4I} \prod_a \frac{d^3 p'_a}{(2\pi)^3 2E'_a},$$

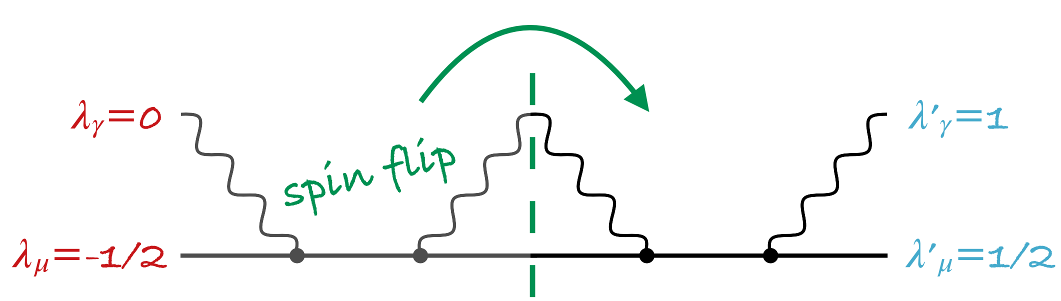
with conserved helicity: $H = \lambda'_\gamma - \lambda'_\mu = \lambda_\gamma - \lambda_\mu$



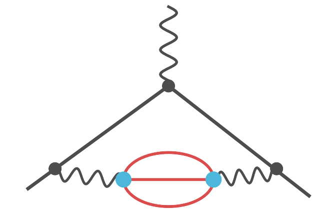
- helicity difference photo-absorption cross section: $\sigma_{TT} = {}^1\!/_2\,(\sigma_{1/2} - \sigma_{3/2})$

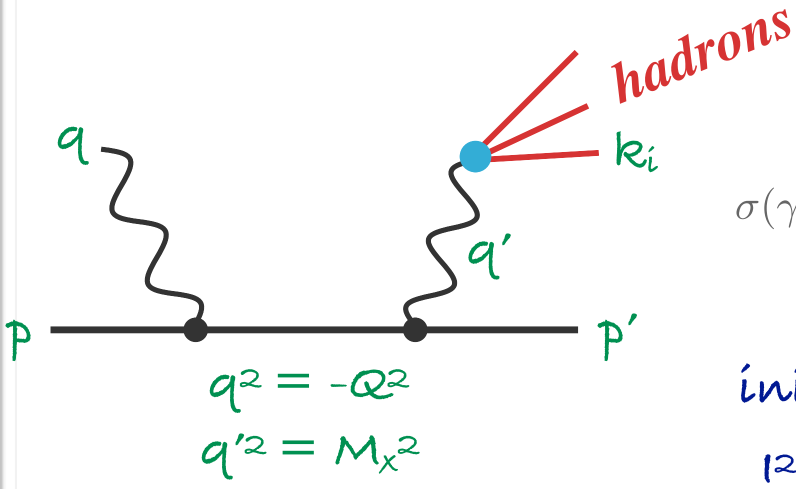
- longitudinal-transverse photo-absorption cross section:

$\gamma^*(\lambda_\gamma=0) + \mu(\lambda_\mu=-1/2) \rightarrow \gamma(\lambda'_\gamma=1) + \mu(\lambda'_\mu=1/2)$



TIME-LIKE CS & PHOTON DECAY





+ crossed diagram

Λ^ν: virtual-photon decay vertex

$$\sigma(\gamma\mu \to \mu X) = \frac{(2\pi)^4}{4I} \int d^4 q' \int \prod_i \frac{d^3 \boldsymbol{k}_i}{2E_{k_i}(2\pi)^3} \int \frac{d^3 \boldsymbol{p'}}{2E_{p'}(2\pi)^3} \left[\frac{\Lambda^{\dagger\mu}\Lambda^\nu \rho_{\mu\nu}}{(-q'^2)^2} \right] \delta^4(q' - \sum_i k_i)\, \delta^4(p + q - p' - q')$$

initial flux factor $I^2 = (p\cdot q)^2 - p^2 q^2$

phase space of the final state

$\rho_{\mu\nu}$: squared matrix element of timelike CS

- Virtual-photon decay width into hadronic state X:

$$[\Gamma(\gamma^* \to X)]^{\mu\nu} = \int \prod_i \frac{d^3 \boldsymbol{k_i}}{2E_{k_i}(2\pi)^3} \frac{\Lambda^{\dagger\mu}\Lambda^\nu}{2E_{q'}} (2\pi)^4 \delta^4(q' - \sum_i k_i)$$

$$= -\frac{1}{\sqrt{q'^2}} (q'^2 g^{\mu\nu} - q'^\mu q'^\nu) \operatorname{Im} \Pi_X(q'^2)$$

$\operatorname{Im} \Pi_X$: contribution of state X to the VP

- Combine into: $\sigma(\gamma\mu \to \mu X) = -\frac{1}{2I} \int d^4 q' \int \frac{d^3 \boldsymbol{p'}}{2E_{p'}(2\pi)^3} \rho^\mu_\mu \frac{\operatorname{Im} \Pi_X(q'^2)}{q'^2} \delta^4(p + q - p' - q')$

- Final factorized cross section:

$$\boxed{\sigma(\gamma\mu \to \mu X) = \frac{1}{\pi} \int_{4m_\pi^2}^\infty \frac{dM_X^2}{M_X^2} \sigma(\gamma\mu \to \gamma^*\mu) \operatorname{Im} \Pi_X(M_X^2)}$$