Beyond one- and two-pion intermediate states

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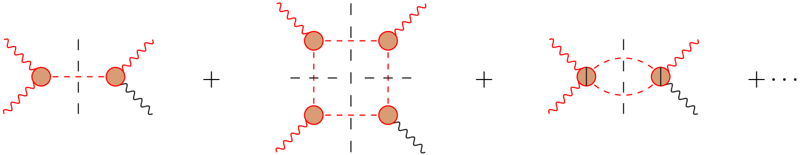
G. Colangelo, MH, M. Procura, P. Stoffer, work in progress



$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

- Organized in terms of **on-shell intermediate states**
- Numerical estimates for $a_\mu^{\pi^0\text{-pole}}$ Talk by B. Kubis and $a_\mu^{\pi\text{-box}}$, $a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}}$ Talk by P. Stoffer
- Other pseudoscalar (η, η') and two-meson states ($K\bar{K}$, $\pi\eta$) implied to be included
- Here: ideas how to treat the ellipsis

BTT decomposition: reminder

$$a_\mu^{\text{HLbL}} = \frac{\alpha^3}{432\pi^2} \int_0^\infty d\Sigma\,\Sigma^3 \int_0^1 dr\, r\sqrt{1-r^2} \int_0^{2\pi} d\phi \sum_{i=1}^{12} T_i(Q_1,Q_2,Q_3)\bar{\Pi}_i(Q_1,Q_2,Q_3)$$

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\Pi_i \qquad \bar{\Pi}_i \text{ linear combination of } \Pi_i \qquad Q_i = Q_i(\Sigma, r, \phi)$$

- **Bardeen–Tung–Tarrach (BTT) decomposition**
- Π_i free of kinematic singularities and zeros

  ↪ dispersive treatment
- A lot of the complexity separated into kernel functions T_i
- Only 6 independent $\bar{\Pi}_i$ thanks to crossing symmetry
- How to constrain the Π_i beyond one- and two-pion intermediate states?

Outline

1. Asymptotic behavior and OPE constraints
2. Narrow resonances
3. A test case: the $f_2(1270)$
4. Conclusions/outlook

Familiar contributions in BTT form

- **Pion pole**

$$\bar{\Pi}_1(q_1^2,q_2^2,q_3^2)=\frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)F_{\pi^0\gamma^*\gamma^*}(q_3^2,0)}{q_3^2-M_{\pi^0}^2} \qquad \bar{\Pi}_2(q_1^2,q_2^2,q_3^2)=\frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2)F_{\pi^0\gamma^*\gamma^*}(q_2^2,0)}{q_2^2-M_{\pi^0}^2}$$

- **Pion loop**

$$\bar{\Pi}_i^{\pi\text{-box}}(q_1^2,q_2^2,q_3^2)=F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\,\bar{I}_i(x,y)$$

$$\bar{I}_1(x,y)=\frac{8xy(1-2x)(1-2y)}{\Delta_{123}\Delta_{23}} \qquad \bar{I}_5(x,y)=-\frac{8xy(1-x-y)(1-2x)^2(1-2y)}{\Delta_{123}^3} \qquad \dots$$

$$\Delta_{ijk}=M_\pi^2-xyq_i^2-x(1-x-y)q_j^2-y(1-x-y)q_k^2 \qquad \Delta_{ij}=M_\pi^2-x(1-x)q_i^2-y(1-y)q_j^2$$

- BTT decomposition **isolates the dynamical content**, separates the kinematics

  $\hookrightarrow$ do the same for the fermion loop

Fermion loop in BTT decomposition

- **Fermion loop**

$$\bar{\Pi}_i^{f\text{-loop}}(q_1^2, q_2^2, q_3^2) = N_c Q_f^4 \frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy\, \bar{I}_i(x, y)$$

$$\bar{I}_1(x, y) = -\frac{16x(1-x-y)}{\Delta_{132}^2} - \frac{16xy(1-2x)(1-2y)}{\Delta_{132}\Delta_{32}} \qquad \bar{I}_5(x, y) = -\frac{64xy^2(1-x-y)(1-2x)(1-y)}{\Delta_{132}^3}$$

$$\Delta_{ijk} = m_f^2 - xyq_i^2 - x(1-x-y)q_j^2 - y(1-x-y)q_k^2 \qquad \Delta_{ij} = m_f^2 - x(1-x)q_i^2 - y(1-y)q_j^2$$

- Should provide constraints on **asymptotic behavior** of the Π_i
- Numerical cross checks

f	e	μ	τ	c	b
$a_\mu^{f\text{-loop}}$ $[10^{-11}]$	26257(3)	464.97(5)	2.686(3)	3.038(3)	0.018(3)
Jegerlehner, Nyffeler 2009	26253.5102(2)	464.971652	2.68556(86)		
Asymptotic expansion, Kühn et al. 2003				3.04	0.0182

OPE and triangle amplitude

- OPE for $q_1^2 \sim q_2^2 \gg q_3^2$ Melnikov, Vainshtein 2004

$$i\int d^4x\, d^4y\, e^{-iq_1\cdot x - iq_2\cdot y}\, T\{j_\mu(x), j_\nu(y)\} = \int d^4z\, e^{-i(q_1+q_2)\cdot z}\, \frac{2i}{\hat{q}^2}\epsilon_{\mu\nu\lambda\sigma}\hat{q}^\lambda j_5^\sigma(z) + \cdots$$

$$j^\mu = \bar{q}Q\gamma^\mu q \qquad j_5^\mu = \bar{q}Q^2\gamma^\mu\gamma_5 q \qquad \hat{q} = \frac{q_1 - q_2}{2} \qquad Q = \frac{e}{3}\text{diag}(2, -1, -1)$$

- **Non-renormalization theorems** for VVA triangle (in chiral limit), c.f. a_μ^{EW}
  Czarnecki, Marciano, Vainshtein 2003, Knecht, Peris, Perrottet, de Rafael 2002, 2004, Mondejar, Melnikov 2013
- Proposed interpolation between ABJ anomaly and asymptotic behavior

$$\bar{\Pi}_1^{\text{MV}}(q_1^2, q_2^2, q_3^2) = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\boldsymbol{F_{\pi^0\gamma^*\gamma^*}(0,0)}}{q_3^2 - M_{\pi^0}^2} \qquad \bar{\Pi}_2^{\text{MV}}(q_1^2, q_2^2, q_3^2) = \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_3^2)\boldsymbol{F_{\pi^0\gamma^*\gamma^*}(0,0)}}{q_2^2 - M_{\pi^0}^2}$$

$\hookrightarrow$ **disturbs low-energy properties**, at what scale should $1/q^2$ behavior set in?

- Model constructed in Melnikov, Vainshtein 2004: frequent reference to asymptotics of pQCD quark loop

  $\hookrightarrow$ make connection directly at the level of BTT functions!

OPE and Brodsky–Lepage limit

- Separation into **hard scattering kernel** and **meson distribution amplitudes**
  Brodsky, Lepage 1979, 1980, 1981

- Simplest case: pion transition form factor

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{2e^2 F_\pi}{3}\int_0^1 \mathrm{d}x \frac{\phi_\pi(x)}{xq_1^2 + (1-x)q_2^2}$$

- Relation to OPE Manohar 1990: only strictly justified for $\omega = 2\frac{q_1^2 - q_2^2}{q_1^2 + q_2^2} < 1$
- **Brodsky–Lepage limit**

$$F_{\pi^0\gamma^*\gamma^*}(-Q^2, 0) = \frac{2e^2 F_\pi}{Q^2}$$

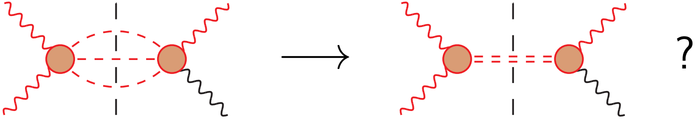
  amounts to resummation of OPE

- Constraints on $\gamma\gamma \to \pi\pi$ Brodsky, Lepage 1981 useful for asymptotic behavior?

- OPE for $\gamma^*\gamma^* \to \pi\pi$ Bijnens, Relefors 2016

$$\lim_{Q^2\to\infty} A\big(\gamma^*(q_1 = Q + k)\gamma^*(q_2 = -Q + k) \to \pi(p_1)\pi(p_2)\big) \sim \frac{1}{Q^2}$$

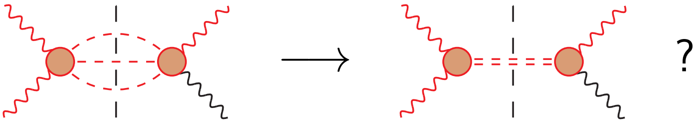
Narrow resonances as estimates for higher intermediate states



- Idea: 3π, 4π, ... probably dominated by resonant states

  $\hookrightarrow$ can one approximate these channels by **narrow resonances** $\delta(s - m_R^2)$?
- **Practical issues**:
  - Many resonances are not that narrow
  - Scarce data on required form factors, in particular doubly-virtual ones
- **Conceptual issues**:
  - Lagrangian-based formulation does not give the residue except for pseudoscalars
  - BTT ambiguities only absent for pseudoscalars

Narrow resonances as estimates for higher intermediate states



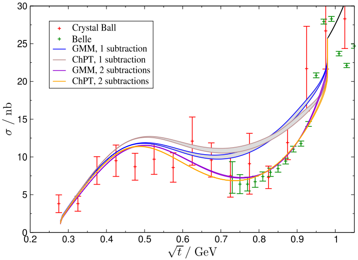
- **Dispersive approach**:
  - Derived BTT form factor decomposition for scalar, axial, and tensor resonances
  - Implemented HLbL contribution in terms of BTT form factors
- But: does not solve the practical issues
- **Perfect test case: $f_2(1270)$**
  - Decays predominantly to $\pi\pi \Rightarrow$ captured by rescattering corrections
  - Not too broad $\Rightarrow$ narrow-width approximation not unreasonable

$\hookrightarrow$ compare both descriptions

Roy–Steiner equations for $\gamma^*\gamma^* \to \pi\pi$

Roy(–Steiner) equations $=$ **Dispersion relations** $+$ **partial-wave expansion** $+$ **crossing symmetry** $+$ **unitarity** $+$ **gauge invariance**

- $f_2(1270)$ strong D-wave resonance in $\pi\pi$ scattering
  $\hookrightarrow$ similar to rescattering corrections in S-wave
- **On-shell case** $\gamma\gamma \to \pi\pi$ García-Martín, Moussallam 2010, MH, Phillips, Schat 2011, partial-wave analysis Dai, Pennington 2014
- **Singly-virtual** $\gamma^*\gamma \to \pi\pi$ Moussallam 2013
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: new technical challenges Colangelo et al. 2013, 2015, 2017
  - anomalous thresholds (only for time-like kinematics)
  - coupling of different partial waves important



Example from 1-loop ChPT

- At 1-loop only S-waves for $\gamma^*\gamma^* \to \pi\pi$ (apart from Born terms)

$$h_{0,++}(s;q_1^2,q_2^2) = \frac{\bar{l}_6-\bar{l}_5}{48\pi^2F_\pi^2}(s-q_1^2-q_2^2)\begin{Bmatrix}1\\0\end{Bmatrix} - \frac{1}{8\pi^2F_\pi^2}\begin{Bmatrix}s/2\\s-M_\pi^2\end{Bmatrix}\bigg\{1+2\Big(M_\pi^2+\frac{sq_1^2q_2^2}{\lambda_{12}(s)}\Big)C_0(s,q_1^2,q_2^2)$$
$$+\frac{s(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda_{12}(s)}\bar{J}(s) - \frac{q_1^2(s+q_2^2-q_1^2)}{\lambda_{12}(s)}\bar{J}(q_1^2) - \frac{q_2^2(s+q_1^2-q_2^2)}{\lambda_{12}(s)}\bar{J}(q_2^2)\bigg\}$$
$$\bar{h}_{0,00}(s;q_1^2,q_2^2) = \frac{\bar{l}_6-\bar{l}_5}{24\pi^2F_\pi^2}\begin{Bmatrix}1\\0\end{Bmatrix} + \frac{1}{8\pi^2F_\pi^2\lambda_{12}(s)}\begin{Bmatrix}s/2\\s-M_\pi^2\end{Bmatrix}\bigg\{\Big(s^2-(q_1^2-q_2^2)^2\Big)C_0(s,q_1^2,q_2^2)$$
$$+4s\bar{J}(s) - 2(s+q_1^2-q_2^2)\bar{J}(q_1^2) - 2(s-q_1^2+q_2^2)\bar{J}(q_2^2)\bigg\}$$

- Fulfill dispersion relation

$$\frac{1}{\pi}\int_{4M_\pi^2}^\infty ds'\bigg\{\Big(\frac{1}{s'-s}-\frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')}\Big)\text{Im}\,h_{0,++}(s';q_1^2,q_2^2) + \frac{2q_1^2q_2^2}{\lambda_{12}(s')}\text{Im}\,\bar{h}_{0,00}(s';q_1^2,q_2^2)\bigg\}$$
$$= 1+2\Big(M_\pi^2+\frac{sq_1^2q_2^2}{\lambda_{12}(s)}\Big)C_0(s,q_1^2,q_2^2) + \frac{s(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda_{12}(s)}\bar{J}(s) - \frac{q_1^2(s+q_2^2-q_1^2)}{\lambda_{12}(s)}\bar{J}(q_1^2) - \frac{q_2^2(s+q_1^2-q_2^2)}{\lambda_{12}(s)}\bar{J}(q_2^2)$$
$$\frac{1}{\pi}\int_{4M_\pi^2}^\infty ds'\bigg\{\Big(\frac{1}{s'-s}-\frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')}\Big)\text{Im}\,\bar{h}_{0,00}(s';q_1^2,q_2^2) + \frac{2}{\lambda_{12}(s')}\text{Im}\,h_{0,++}(s';q_1^2,q_2^2)\bigg\}$$
$$= -\frac{1}{\lambda_{12}(s)}\bigg\{\Big(s^2-(q_1^2-q_2^2)^2\Big)C_0(s,q_1^2,q_2^2) + 4s\bar{J}(s) - 2(s+q_1^2-q_2^2)\bar{J}(q_1^2) - 2(s-q_1^2+q_2^2)\bar{J}(q_2^2)\bigg\}$$

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$$=1+2\bigg(M_\pi^2+\frac{sq_1^2q_2^2}{\lambda_{12}(s)}\bigg)C_0(s,q_1^2,q_2^2)+\frac{s(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda_{12}(s)}\bar J(s)-\frac{q_1^2(s+q_2^2-q_1^2)}{\lambda_{12}(s)}\bar J(q_1^2)-\frac{q_2^2(s+q_1^2-q_2^2)}{\lambda_{12}(s)}\bar J(q_2^2)$$

$$\frac{1}{\pi}\int_{4M_\pi^2}^\infty ds'\bigg\{\bigg(\frac{1}{s'-s}-\frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')}\bigg)\text{Im}\,\bar h_{0,00}(s';q_1^2,q_2^2)+\frac{2}{\lambda_{12}(s')}\text{Im}\, h_{0,++}(s';q_1^2,q_2^2)\bigg\}$$

$$=-\frac{1}{\lambda_{12}(s)}\bigg\{\Big(s^2-(q_1^2-q_2^2)^2\Big)C_0(s,q_1^2,q_2^2)+4s\bar J(s)-2(s+q_1^2-q_2^2)\bar J(q_1^2)-2(s-q_1^2+q_2^2)\bar J(q_2^2)\bigg\}$$

- Off-diagonal contributions critical, related to avoidance of **kinematic singularities**

- How to find these kernel functions, in general?

  $\hookrightarrow$ start from **BTT decomposition** of $\gamma^*\gamma^*\to\pi\pi$!

BTT decomposition of $\gamma^*\gamma^* \to \pi\pi$

$$H_{++} = -\frac{1}{2}(s - q_1^2 - q_2^2)A_1 - q_1^2 q_2^2 A_2 + \frac{1}{2s}(s - 4M_\pi^2)\lambda_{12}(s) z^2 (q_1^2 + q_2^2) A_3$$
$$+ \frac{1}{4}(s - 4M_\pi^2)\left((s - q_1^2 - q_2^2) + \left(\frac{(q_1^2 - q_2^2)^2}{s} - (q_1^2 + q_2^2)\right) z^2\right) A_4 + \frac{1}{2} q_1^2 q_2^2 (s - 4M_\pi^2)(1 - z^2) A_5$$
$$H_{+-} = -\frac{1}{4}(s - 4M_\pi^2)(1 - z^2)\Big((s - q_1^2 - q_2^2)A_4 + 2 q_1^2 q_2^2 A_5\Big)$$
$$\bar{H}_{+0} = \frac{1}{4}\sqrt{\frac{2}{s}}(s - 4M_\pi^2) z \sqrt{1 - z^2}\Big(\lambda_{12}(s) A_3 - (s + q_1^2 - q_2^2) A_4 - q_1^2 (s - q_1^2 + q_2^2) A_5\Big)$$
$$\bar{H}_{0+} = \frac{1}{4}\sqrt{\frac{2}{s}}(s - 4M_\pi^2) z \sqrt{1 - z^2}\Big(\lambda_{12}(s) A_3 - (s - q_1^2 + q_2^2) A_4 - q_2^2 (s + q_1^2 - q_2^2) A_5\Big)$$
$$\bar{H}_{00} = -A_1 - \frac{1}{2}(s - q_1^2 - q_2^2) A_2 - \frac{1}{s}(s - 4M_\pi^2)\lambda_{12}(s) z^2 A_3 + (s - 4M_\pi^2) z^2 A_4 + \frac{1}{4s}(s - 4M_\pi^2)\left(s^2 - (q_1^2 - q_2^2)^2\right) z^2 A_5\Big)$$

- Decomposition of helicity amplitudes in terms of BTT scalar functions A_i
  ↪ the A_i **fulfill simple dispersion relations**, not the helicity partial waves!
- For S-waves: only A_1 and A_2 relevant Talk by P. Stoffer
- **For $f_2(1270)$**: main effect expected in H_{+-}
  ↪ need at least A_4 and A_5 ⇒ **induced S-waves**

Omnès solution

- Solution for ***S*-waves**

$$h_{0,++}(s) = \Delta_{0,++}(s) + \frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')}{|\Omega_0(s')|}\bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')}\bigg)\Delta_{0,++}(s') + \frac{2q_1^2q_2^2}{\lambda_{12}(s')}\Delta_{0,00}(s')\bigg]$$

$$h_{0,00}(s) = \Delta_{0,00}(s) + \frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')}{|\Omega_0(s')|}\bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')}\bigg)\Delta_{0,00}(s') + \frac{2}{\lambda_{12}(s')}\Delta_{0,++}(s')\bigg]$$

$\hookrightarrow$ note the same kernel functions as before!

- $\pi\pi$ S-wave phase shift $\delta_0(s)$ defines Omnès factor $\Omega_0(s) = \exp\Big\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\delta_0(s')}{s'(s'-s)}\Big\}$
- Left-hand cut included in $\Delta_{0,++}(s)$, $\Delta_{0,00}(s)$
- Amazingly, a very similar system emerges if only A_4 and A_5 are kept!

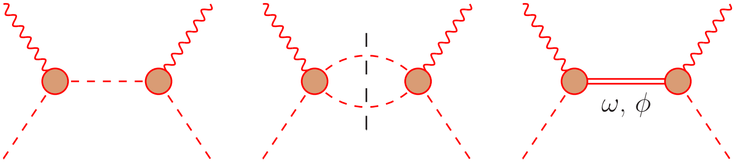
$$h_{2,+-}(s) = \Delta_{2,+-}(s) + \frac{\Omega_2(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{s-4M_\pi^2}{s'-4M_\pi^2}\frac{\sin\delta_2(s')}{|\Omega_2(s')|}\bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')}\bigg)\Delta_{2,+-}(s') + \frac{2q_1^2q_2^2}{\lambda_{12}(s')}\tilde{\Delta}_{2,+0}(s')\bigg]$$

$$\tilde{h}_{2,+0}(s) = \tilde{\Delta}_{2,+0}(s) + \frac{\Omega_2(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{s-4M_\pi^2}{s'-4M_\pi^2}\frac{\sin\delta_2(s')}{|\Omega_2(s')|}\bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda_{12}(s')}\bigg)\tilde{\Delta}_{2,+0}(s') + \frac{2}{\lambda_{12}(s')}\Delta_{2,+-}(s')\bigg]$$

where $\tilde{h}_{2,+0}(s) = -\frac{\sqrt{2s}}{q_1^2-q_2^2}\big(h_{2,+0}(s) - h_{2,+0}(s)\big)$

- Next: how to determine the left-hand cut

Left-hand cut



- **Pion pole**: coupling determined by pion vector form factor $F_\pi^V(s)$

  $\hookrightarrow$ dominant effect for S-waves
- **Multi-pion intermediate states**: approximate in terms of **resonances**
  - $2\pi \sim \rho$: can even be done **exactly** using $\gamma^* \to 3\pi$ amplitude Talk by B. Kubis
  - $3\pi \sim \omega, \phi$: narrow-width approximation clearly justified

    $\hookrightarrow$ **transition form factors** for $\omega, \phi \to \pi^0\gamma^*$ Schneider, Kubis, Niecknig 2012
- Cannot reproduce the $f_2(1270)$ in $\gamma\gamma \to \pi\pi$ without the **vector resonances**

  García-Martín, Moussallam 2010

Role of pion polarizabilities

- **Low-energy expansion**

$$h_{2,+-}(s) = h^{\text{Born}}_{2,+-}(s) + \frac{s(s-4M_\pi^2)}{10\sqrt{6}M_\pi\alpha}(\alpha_1+\beta_1) + \cdots$$

$\hookrightarrow$ sum rule for the **pion polarizability** $\alpha_1+\beta_1$

- Compare **photon couplings**

$$\frac{\Gamma_\omega \text{Br}[\omega\to\pi^0\gamma]}{\Gamma_\rho \text{Br}[\rho^{\pm,0}\to\pi^{\pm,0}\gamma]} \sim 10$$

$\hookrightarrow$ if vector resonances dominate, expect $(\alpha_1+\beta_1)^{\pi^0} \gg (\alpha_1+\beta_1)^{\pi^\pm}$

Role of pion polarizabilities

- **Low-energy expansion**

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$\hookrightarrow$ if vector resonances dominate, expect $(\alpha_1+\beta_1)^{\pi^0} \gg (\alpha_1+\beta_1)^{\pi^\pm}$

- Indeed:
  - **Two-loop ChPT** Gasser, Ivanov, Sainio 2005, 2006

$$(\alpha_1+\beta_1)^{\pi^0} = 1.1(3)\times 10^{-4}\text{fm}^3 \qquad (\alpha_1+\beta_1)^{\pi^\pm} = 0.16\times 10^{-4}\text{fm}^3$$

  - In agreement with fit to $\gamma\gamma \to \pi\pi$ cross section García-Martín, Moussallam 2010

- Status of the $f_2(1270)$: preliminary results indicate that the induced S-waves may actually be numerically dominant for a_μ^{HLbL}

Conclusions/outlook

- **Asymptotic behavior:**
  - pQCD quark loop in BTT basis
  - additional constraints from VVA non-renormalization theorems?
  - more to be learned from OPE of the HLbL tensor?
- **Multi-particle intermediate states beyond two mesons**
  - narrow resonances might give reasonable estimates
  - needs to be formulated dispersively to be consistent
- **$f_2(1270)$ as test case**
  - requires extended input for $\gamma^*\gamma^* \to \pi\pi$ partial waves
  - numerical analysis in progress