Hadronic light-by-light scattering contribution to the muon $g-2$ on the lattice: overall strategy

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Outline

- Status of the muon $g-2$
- Status of hadronic light-by-light (HLbL) scattering in the muon $g-2$
- HLbL on the lattice: activities by Mainz group
- Position space approach to HLbL on the lattice, master formula, QED kernel in the continuum and in infinite volume
- Numerical tests of QED kernel
  - (I) Pion-pole contribution
  - (II) a_μ^{LbL} from lepton loop in QED
- Conclusions and Outlook

Muon $g-2$: current status

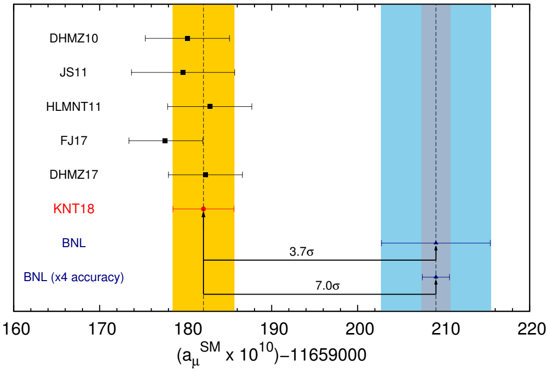
Contribution	$a_\mu \times 10^{11}$	Reference
QED (leptons)	116 584 718.853 $\pm$ 0.036	Aoyama et al. '12
Electroweak	153.6 $\pm$ 1.0	Gnendiger et al. '13
HVP: LO	6887.7 $\pm$ 33.8	Jegerlehner '17
NLO	-99.3 $\pm$ 0.7	Jegerlehner '17
NNLO	12.4 $\pm$ 0.1	Kurz et al. '14
HLbL	102 $\pm$ 39	Jegerlehner '15 (JN '09)
NLO	3 $\pm$ 2	Colangelo et al. '14
Theory (SM)	116 591 778 $\pm$ 52	
Experiment	116 592 089 $\pm$ 63	Bennett et al. '06
Experiment - Theory	311 $\pm$ 81	3.8 σ

HLbL based on Jegerlehner, AN '09, with downward shift because of smaller axial-vector contribution (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). Other frequently used estimate for HLbL: $a_\mu^{\text{HLbL}} = (105 \pm 26) \times 10^{-11}$ (Prades, de Rafael, Vainshtein '09 ("Glasgow consensus")).

Discrepancy a sign of New Physics ?

Hadronic uncertainties need to be better controlled in order to fully profit from future $g-2$ experiments at Fermilab (E989) and J-PARC (E34) with four-fold improvement $\delta a_\mu = 16 \times 10^{-11}$.

Muon $g-2$: other recent theoretical evaluations



3.6 σ

3.3 σ

3.3 σ

4.1 σ (includes τ)

3.5 σ

Keshavarzi, Nomura, Teubner '18, arXiv:1802.02995

Note units of 10^{-10} !

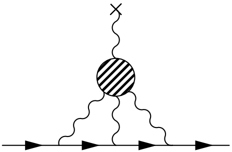
Estimates for HVP contribution:

FJ17 ($e^+e^-[+\tau]$): $a_\mu^{\text{LO HVP}} = (6881 \pm 41) \times 10^{-11}$ $\quad[(6888 \pm 34) \times 10^{-11}]$

DHMZ17: $a_\mu^{\text{LO HVP}} = (6931 \pm 34) \times 10^{-11}$

KNT18: $a_\mu^{\text{LO HVP}} = (6933 \pm 25) \times 10^{-11}$

Status of HLbL in the muon $g-2$



- Not fully related to experimental cross sections.
- Until now only model dependent estimates of total contribution (Prades, de Rafael, Vainsthein '09; Jegerlehner, AN '09; Jegerlehner '15, '17).
- Large model uncertainties.
- Uncertainties need to be reduced and better controlled.
- Phenomenology
  Use dispersion relations to reduce model uncertainties for dominant contributions using experimental input from
  $\gamma^* \gamma^{(*)} \to \pi^0, \eta, \eta'; \pi^+\pi^-, \pi^0\pi^0$ (still to be measured !)
  Try to reconstruct input for DR from other hadronic processes.
  Colangelo *et al.* '14, '15, '17
  Pauk and Vanderhaeghen '14. More recently combined with HLbL forward scattering sum rules (Pascalutsa, Pauk, Vanderhaeghen '12): Danilkin and Vanderhaeghen '17
  Recent proposal: Schwinger's sum rule (Hagelstein and Pascalutsa '17)
- Lattice QCD
  Can provide a model independent first-principle estimate.
  Only publications from essentially one group so far: Blum *et al.* (RBC-UKQCD) '05, ..., '15, '16, '17

HLbL on the lattice: activities by Mainz group

Position-space approach to HLbL scattering in the muon $g-2$ on the lattice

Nils Asmussen, Antoine Gérardin, Jeremy Green, Harvey Meyer, AN

First presented in talks by Nils Asmussen at meeting of German Physical Society (DPG), Heidelberg, March 2015 and by Jeremy Green at Lattice 2015 (arXiv:1510.08384). More details in talks by Nils Asmussen at Lattice 2016 (arXiv:1609.08454), Lattice 2017 (arXiv:1711.02466) and by Harvey Meyer at first "Muon $g-2$ Theory Initiative" Workshop near Fermilab 2017.
This talk: overall strategy of the Mainz approach.
New results: see talk by Nils Asmussen

Lattice calculation of the pion transition form factor $\pi^0 \to \gamma^*\gamma^*$

Antoine Gérardin, Harvey Meyer, AN

Phys. Rev. **D94**, 074507 (2016). Update: talk by Antoine Gérardin
Might help to better control finite volume effects from pion-pole contribution.

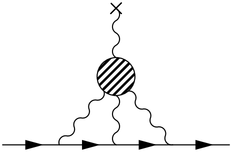
Light-by-light forward scattering amplitudes in lattice QCD

Antoine Gérardin, Jeremy Green, Oleksii Gryniuk, Georg von Hippel, Harvey Meyer, Vladimir Pascalutsa, Hartmut Wittig

Phys. Rev. Lett. **115**, 222003 (2015), Gérardin at al., arXiv:1712.00421,
Talk by Antoine Gérardin
Test of phenomenological approaches and different lattice calculations by comparing several functions of the invariants, not just one number a_μ^{HLbL}.

HLbL in muon $g-2$ from Lattice QCD: Mainz approach

Developed independently from RBC-UKQCD
(Asmussen, Gérardin, Green, Meyer, AN '15 – '17)



- QCD blob: lattice regularization
- Everything else: position-space perturbation theory in Euclidean formulation

Similarities to approach by RBC-UKQCD '15, '16, '17:

- Position-space (most natural for lattice QCD)
- Perturbative treatment of the QED part
- Get directly $a_\mu^{\text{HLbL}} = F_2(k^2 = 0)$ as spatial moment

Strengths of our approach:

- Semi-analytical calculation
- QED part computed in continuum and in infinite volume
- Lorentz covariance manifest
- No power law effects $1/L^2$ in the volume

Challenges:

- Need to calculate a QCD four-point function on the lattice
- Numerical efficiency not yet shown

Focus in this talk on semi-analytical calculation of perturbative QED part.

Vertex function for HLbL in momentum space

In Euclidean space ($k = p' - p$):

$$\langle \mu^-(p', s')| j_\rho(0) |\mu^-(p, s)\rangle = -\bar{u}^{s'}(p') \left[\gamma_\rho F_1(k^2) + \frac{\sigma_{\rho\tau} k_\tau}{2m} F_2(k^2)\right] u^s(p)$$

Project on anomalous magnetic moment:

$$a_\mu^{\text{HLbL}} = F_2(0) = \frac{-i}{48m} \text{Tr}\{[\gamma_\rho, \gamma_\sigma](-i\not{p} + m)\Gamma_{\rho\sigma}(p, p)(-i\not{p} + m)\}$$

with on-shell muon momentum $p = im\hat{\epsilon}$ ($p^2 = -m^2$; $\hat{\epsilon}$: unit vector).

$$\begin{aligned}\Gamma_{\rho\sigma}(p', p) = -\,e^6 \int_{q_1, q_2} & \frac{1}{q_1^2 q_2^2 (q_1+q_2-k)^2} \frac{1}{(p'-q_1)^2+m^2} \frac{1}{(p'-q_1-q_2)^2+m^2} \\ & \times \gamma_\mu (i\not{p}' - i\not{q}_1 - m)\gamma_\nu (i\not{p} - i\not{q}_1 - i\not{q}_2 - m)\gamma_\lambda \\ & \times \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)\end{aligned}$$

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \int_{x_1, x_2, x_3} e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \langle j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\sigma(0)\rangle$$

Where we used the following relation derived from the Ward identities to extract one factor of k to get $F_2(k^2)$ [Kinoshita *et al.* '70]:

$$\Pi_{\mu\nu\lambda\rho}(q_1, q_2, k - q_1 - q_2) = -k_\sigma \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$$

Notation: $\int_q \equiv \int \frac{\mathrm{d}^4 q}{(2\pi)^4}$, $\int_x \equiv \int \mathrm{d}^4 x$

HLbL in muon $g-2$ in position-space

Vertex function in terms of position-space functions:

$$\Gamma_{\rho\sigma}(p,p) = -\, e^6 \int_{x,y} K_{\mu\nu\lambda}(x,y,p)\, \widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)$$

$$K_{\mu\nu\lambda}(x,y,p) = \gamma_\mu (i\not{p} + \not{\partial}^{(x)} - m)\gamma_\nu (i\not{p} + \not{\partial}^{(x)} + \not{\partial}^{(y)} - m)\gamma_\lambda\, \mathcal{I}(\hat{\epsilon},x,y)$$

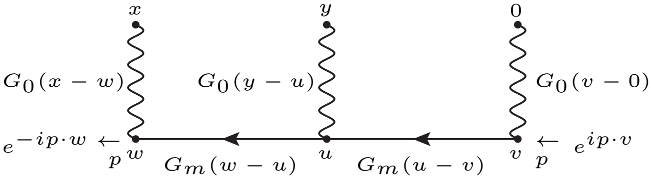
$$\mathcal{I}(\hat{\epsilon},x,y) = \int_{q,k,\text{IR-reg}} \frac{1}{q^2 k^2 (q+k)^2} \frac{1}{(p-q)^2+m^2} \frac{1}{(p-q-k)^2+m^2}\, e^{-i(q\cdot x + k\cdot y)}$$

$$\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \int_z i z_\rho \,\langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0)\rangle$$

- $\mathcal{I}$ is logarithmically infrared divergent for $p^2 = -m^2$
  $\Rightarrow$ introduce IR regulator.
- In a_μ^{HLbL} only terms with derivatives remain and $K_{\mu\nu\lambda}$ is infrared finite.

Evaluating $\mathcal{I}(\hat{\epsilon}, x, y)$

Diagrammatic representation of $\mathcal{I}(\hat{\epsilon}, x, y)$ (arrows on muon line are only a reminder of the origin of the diagram, G_m are scalar propagators !):



$$\mathcal{I}(\hat{\epsilon}, x, y) = \int_{u,\text{IR-reg}} G_0(u-y) J(\hat{\epsilon}, u) J(\hat{\epsilon}, x-u)$$

$$J(\hat{\epsilon}, u) = \int_v G_0(v+u) e^{-m\hat{\epsilon}\cdot v} G_m(v) = \sum_{n\geq 0} z_n(u^2) U_n(\hat{\epsilon}\cdot\hat{u})$$

Last expression: expansion in terms of Chebyshev polynomials of the second kind U_n (special case of the Gegenbauer polynomials)
z_n: linear combination of products of two modified Bessel functions K_m and I_k

Scalar propagators in position-space:

$$G_0(x) = \frac{1}{4\pi^2 x^2}$$

$$G_m(x) = \frac{m}{4\pi^2 |x|} K_1(m|x|) \quad (K_1 \text{ is a modified Bessel function})$$

Averaging over direction of muon momentum $p = im\hat{\epsilon}$

Evaluating Dirac trace in projector, one obtains an expression of the form

$$a_\mu^{\text{HLbL}} = \frac{me^6}{3} \int_y \int_x \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x, y)$$

Exploit invariance of a_μ under $O(4)$ rotations of the muon momentum and average kernel $\mathcal{L}$ over direction $\hat{\epsilon}$ (Barbieri + Remiddi '75)

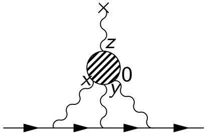
$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y) = \frac{1}{2\pi^2} \int \mathrm{d}\Omega_{\hat{\epsilon}} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) \equiv \langle \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) \rangle_{\hat{\epsilon}}$$

Angular average can be performed analytically by using orthogonality property of Chebyshev (Gegenbauer) polynomials that appear in QED kernel $\mathcal{L}$ via $\mathcal{I}$ and J (hyperspherical approach):

$$\langle U_n(\hat{\epsilon} \cdot \hat{x}) U_m(\hat{\epsilon} \cdot \hat{y}) \rangle_{\hat{\epsilon}} = \frac{\delta_{nm}}{n+1} U_n(\hat{x} \cdot \hat{y})$$

HLbL master formula in position-space

$$a_\mu^{\text{HLbL}} = \frac{me^6}{3} \int d^4y \int d^4x \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\text{QED}} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\text{QCD}}$$



After contracting the Lorentz indices the integration reduces to a 3-dimensional integral over $x^2, y^2, x \cdot y = |x||y| \cos\beta$:

$$a_\mu^{\text{HLbL}} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2 \int_0^\infty d|y||y|^3} \underbrace{\int d^4x}_{=4\pi \int_0^\infty d|x||x|^3 \int_0^\pi d\beta \sin^2(\beta)} \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\text{QED}} \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\text{QCD}}$$

QCD four-point function (spatial moment):

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \, \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$

QED kernel function $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$

- Weights the QCD four-point function in position-space.
- Tensor decomposition leads to 6 weight functions (and derivatives thereof) that depend on the 3 variables $x^2, y^2, x \cdot y$.
- We have computed these weight functions on a grid to about 5 digits precision, once and for all, and stored on disk.

Tensor decomposition of QED kernel and weight functions

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \sum_{A=I,II,III} \mathcal{G}^{A}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} T^{A}_{\alpha\beta\delta}(x,y)$$

$$\mathcal{G}^{I,II,III}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} = \text{sums of products of Kronecker deltas (from Dirac trace)}$$

$$T^{I}_{\alpha\beta\delta}(x,y) = \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta}) V_{\delta}(x,y)$$

$$T^{II}_{\alpha\beta\delta}(x,y) = m\partial^{(x)}_{\alpha}\Big(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta} S(x,y)\Big)$$

$$T^{III}_{\alpha\beta\delta}(x,y) = m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})\Big(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta} S(x,y)\Big)$$

Scalar: $S(x,y) = \langle \mathcal{I} \rangle_{\hat{\epsilon}}$ (IR regulated)

Vector: $V_{\delta}(x,y) = \langle \hat{\epsilon}_{\delta} \mathcal{I} \rangle_{\hat{\epsilon}}$

Tensor: $T_{\beta\delta}(x,y) = \langle (\hat{\epsilon}_{\beta}\hat{\epsilon}_{\delta} - \frac{1}{4}\delta_{\beta\delta}) \mathcal{I} \rangle_{\hat{\epsilon}}$

$$S(x,y) = g^{(0)}$$

$$V_{\delta}(x,y) = x_{\delta} g^{(1)} + y_{\delta} g^{(2)}$$

$$T_{\alpha\beta}(x,y) = \Big(x_{\alpha}x_{\beta} - \frac{x^2}{4}\delta_{\alpha\beta}\Big) l^{(1)} + \Big(y_{\alpha}y_{\beta} - \frac{y^2}{4}\delta_{\alpha\beta}\Big) l^{(2)} + \Big(x_{\alpha}y_{\beta} + y_{\alpha}x_{\beta} - \frac{x\cdot y}{2}\delta_{\alpha\beta}\Big) l^{(3)}$$

where the 6 weight functions depend on $x^2, y^2, x\cdot y$.

Example: Weight function $g^{(2)}(x^2, x\cdot y, y^2)$

$$g^{(2)}(x^2, x\cdot y, y^2) = \frac{1}{8\pi y^2|x|\sin^3\beta}\int_0^\infty du\, u^2 \int_0^\pi d\phi_1$$

$$\times\left[2\sin\beta + \left(\frac{y^2+u^2}{2|u||y|} - \cos\beta\cos\phi_1\right)\frac{\log\chi}{\sin\phi_1}\right]$$

$$\times\sum_{n=0}^{\infty}\Bigg\{ z_n(|u|)z_{n+1}(|x-u|)\left[|x-u|\cos\phi_1\frac{U_n}{n+1} + (|u|\cos\phi_1 - |x|)\frac{U_{n+1}}{n+2}\right]$$

$$+ z_{n+1}(|u|)z_n(|x-u|)\left[(|u|\cos\phi_1 - |x|)\frac{U_n}{n+1} + |x-u|\cos\phi_1\frac{U_{n+1}}{n+2}\right]\Bigg\}$$

where

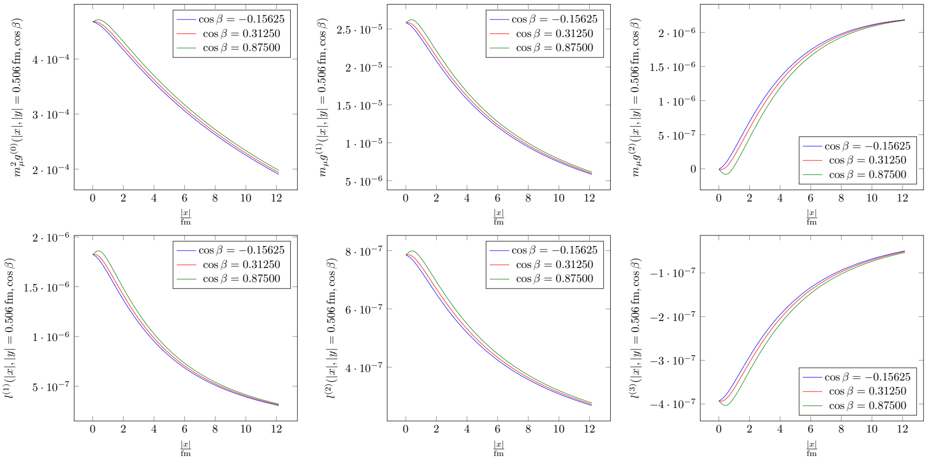
$$x\cdot y = |x||y|\cos\beta\,, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u|\cos\phi_1}$$

$$\chi = \frac{y^2+u^2-2|u||y|\cos(\beta-\phi_1)}{y^2+u^2-2|u||y|\cos(\beta+\phi_1)}\,, \quad U_n = U_n\left(\frac{|x|\cos\phi_1 - |u|}{|u-x|}\right)$$

$z_n =$ linear combination of products of two modified Bessel functions.

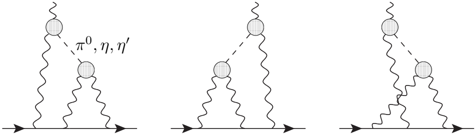
Weight functions: $|x|$ dependence

For $|y| = 0.506$ fm:



$g^{(0)}(|x|, x \cdot y, |y|)$ contains an arbitrary additive constant (due to the IR divergence in $\mathcal{I}(\hat{\epsilon}, x, y)$), which does not contribute to $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x, y)$.

Numerical tests of QED kernel: (I) Pion-pole contribution to a_μ^{HLbL}



Vector-meson dominance (VMD) model for illustration

Pion transition form factor in momentum space:

$$\mathcal{F}^{\text{VMD}}_{\pi^0\gamma^*\gamma^*}(-q_1^2, -q_2^2) = \frac{c_\pi}{(q_1^2 + m_V^2)(q_2^2 + m_V^2)}, \qquad c_\pi = -\frac{N_c m_V^4}{12\pi^2 F_\pi}$$

Parameters in pion-pole contribution: m_V, normalization $1/F_\pi$ and m_π from pion propagator.

HLbL tensor in position-space (convolution):

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = \frac{c_\pi^2}{m_V^2(m_V^2 - m_\pi^2)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y_\beta} \Big\{ \epsilon_{\mu\nu\alpha\beta}\epsilon_{\sigma\lambda\rho\gamma} \Big(\frac{\partial}{\partial x_\gamma} + \frac{\partial}{\partial y_\gamma} \Big) K_\pi(x,y)$$

$$+ \epsilon_{\mu\lambda\alpha\beta}\epsilon_{\nu\sigma\gamma\rho} \frac{\partial}{\partial y_\gamma} K_\pi(y-x, y) + \epsilon_{\mu\sigma\alpha\rho}\epsilon_{\nu\lambda\beta\gamma} \frac{\partial}{\partial x_\gamma} K_\pi(x, x-y) \Big\}$$

where

$$K_\pi(x,y) \equiv \int d^4u \Big(G_{m_\pi}(u) - G_{m_V}(u) \Big) G_{m_V}(x-u) G_{m_V}(y-u) = K_\pi(y,x)$$

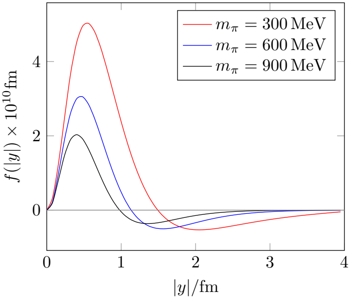
Numerical tests of QED kernel: (I) Pion-pole contribution to a_μ^{HLbL} (cont.)

Result with VMD model for arbitrary pion mass can easily be obtained from 3-dimensional momentum-space representation (Jegerlehner + AN '09).

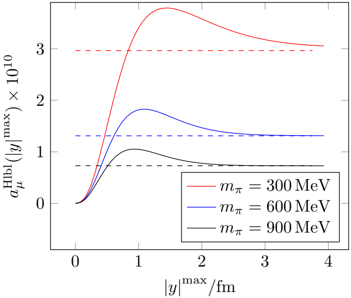
3-dim. integration in position-space:

- $\int_y \to 2\pi^2 \int_0^\infty \mathrm{d}|y||y|^3$
- $\int_x \to 4\pi \int_0^\infty \mathrm{d}|x|\, |x|^3 \int_0^\pi \mathrm{d}\beta \sin^2\beta$ (cutoff for x integration: $|x|^{\max} = 4.05$ fm)

Integrand after integration over $|x|, \beta$:

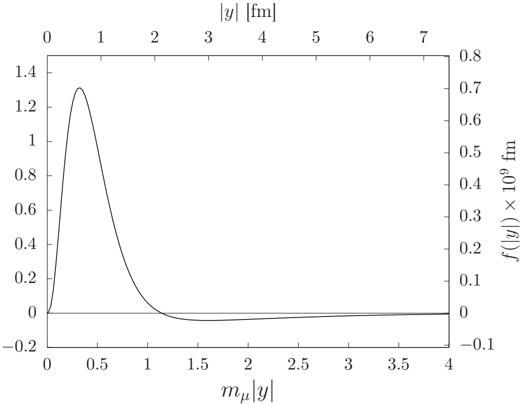


Result for $a_\mu^{\mathrm{HLbL}}(|y|^{\max})$:



- All 6 weight functions contribute to final result, some only at the percent level.
- $|x|^{\max}, |y|^{\max} > 4$ fm needed for $m_\pi < 300$ MeV.

Integrand of pion-pole contribution for physical pion mass



- One needs to go to very large values of $|x|$ and $|y|$, i.e. very large lattice volumes, to reproduce known result 57.0×10^{-11}.
- Result of integration: 57.9×10^{-11}, about 1.6% deviation. The long negative tail is still not fully captured by the extent of the grid and the density of points where we have evaluated the QED kernel.

Numerical tests of QED kernel: (II) a_μ^{LbL} from lepton loop in QED

Fermion propagator in position-space:

$$S_F(x) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{-ip_\mu \gamma_\mu + m}{p^2 + m^2} e^{ipx} = \frac{m^2}{4\pi^2|x|}\left[\gamma_\mu x_\mu \frac{K_2(m|x|)}{|x|} + K_1(m|x|)\right]$$

Analytical result for lepton loop with mass m_l in position space:

$$\begin{aligned} i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) \quad = \quad & \widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) + \widehat{\Pi}^{(1)}_{\rho;\nu\lambda\mu\sigma}(y-x,-x) + \widehat{\Pi}^{(1)}_{\rho;\lambda\nu\mu\sigma}(-x,y-x) \\ & + x_\rho \, \Pi^{(r,1)}_{\nu\lambda\mu\sigma}(y-x,-x) + x_\rho \, \Pi^{(r,1)}_{\lambda\nu\mu\sigma}(-x,y-x) \end{aligned}$$

$$\begin{aligned} \Pi^{(r,1)}_{\mu\nu\lambda\sigma}(x,y) \quad & = 2\left(\frac{m_l}{2\pi}\right)^8 \Big[\frac{(-x_\alpha)(x-y)_\beta \; K_2(m_l|x|)K_2(m_l|x-y|)}{|x|^2|x-y|^2} \cdot I_{\gamma\delta}(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\ & + \frac{K_1(m_l|x|)K_1(m_l|x-y|)}{|x||x-y|} \cdot p(|y|) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\ & + \frac{(-x_\alpha)(x-y)_\beta \; K_2(m_l|x|)K_2(m_l|x-y|)}{|x|^2|x-y|^2} \cdot p(|y|) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\ & + \frac{(-x_\alpha) \; K_2(m_l|x|)K_1(m_l|x-y|)}{|x|^2|x-y|} \cdot q_\gamma(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\ & + \frac{(x-y)_\beta \; K_1(m_l|x|)K_2(m_l|x-y|)}{|x||x-y|^2} \cdot q_\gamma(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\ & + \frac{(-x_\alpha) \; K_2(m_l|x|)K_1(m_l|x-y|)}{|x|^2|x-y|} \cdot q_\delta(y) \cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\ & + \frac{(x-y)_\beta \; K_1(m_l|x|)K_2(m_l|x-y|)}{|x||x-y|^2} \cdot q_\delta(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\ & + \frac{K_1(m_l|x|)K_1(m_l|x-y|)}{|x||x-y|} \cdot I_{\gamma\delta}(y) \cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big] \end{aligned}$$

The lepton loop (continued)

$$\begin{aligned}
\widehat{\Pi}^{(1)}_{\rho;\mu\nu\lambda\sigma}(x,y) \quad &= 2\Big(\frac{m_l}{2\pi}\Big)^8\Big[\frac{(-x_\alpha)(x-y)_\beta K_2(m_l|x|)K_2(m_l|x-y|)}{|x|^2|x-y|^2}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m_l|x|)K_1(m_l|x-y|)}{|x||x-y|}\cdot f_{\rho\delta\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{K_1(m_l|x|)K_1(m_l|x-y|)}{|x||x-y|}\, g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)(x-y)_\beta\ K_2(m_l|x|)K_2(m_l|x-y|)}{|x|^2|x-y|^2} g_\rho(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m_l|x|)K_1(m_l|x-y|)}{|x|^2|x-y|} h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m_l|x|)K_2(m_l|x-y|)}{|x||x-y|^2} h_{\rho\gamma}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\gamma\gamma_\sigma\gamma_\lambda\} \\
&+\frac{(-x_\alpha)\ K_2(m_l|x|)K_1(m_l|x-y|)}{|x|^2|x-y|} \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\alpha\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\} \\
&+\frac{(x-y)_\beta\ K_1(m_l|x|)K_2(m_l|x-y|)}{|x||x-y|^2} \hat{f}_{\rho\delta}(y)\cdot \mathrm{Tr}\{\gamma_\mu\gamma_\beta\gamma_\nu\gamma_\sigma\gamma_\delta\gamma_\lambda\}\Big]
\end{aligned}$$

$$I_{\gamma\delta}(y) = \frac{2\pi^2}{m_l^2}\Big(\hat{y}_\gamma\hat{y}_\delta\ K_2(m_l|y|) - \delta_{\gamma\delta}\ \frac{K_1(m_l|y|)}{m_l|y|}\Big), \quad h_{\rho\gamma}(y) = \frac{\pi^2}{m_l^3}\Big(\hat{y}_\gamma\hat{y}_\rho\ m_l|y|\ K_1(m_l|y|) - \delta_{\gamma\rho}K_0(m_l|y|)\Big),$$

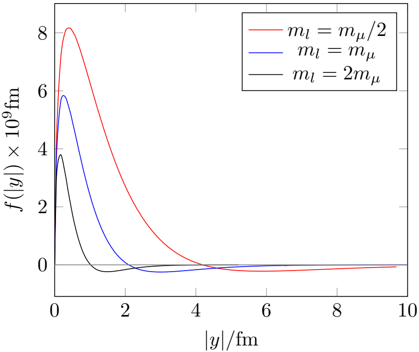
$$\hat{f}_{\rho\delta}(y) = \frac{\pi^2}{m_l^3}\Big\{\hat{y}_\rho\hat{y}_\delta\ m_l|y|K_1(m_l|y|) + \delta_{\rho\delta}K_0(m_l|y|)\Big\}, \qquad g_\rho(y) = \frac{\pi^2}{m_l^2}\ y_\rho\ K_0(m_l|y|), \qquad q_\gamma(y) = \frac{2\pi^2}{m_l^2}\ \hat{y}_\gamma\ K_1(m_l|y|),$$

$$f_{\rho\delta\gamma}(y) = \frac{\pi^2}{m_l^3}\Big\{\hat{y}_\gamma\hat{y}_\delta\hat{y}_\rho\ m_l|y|K_2(m_l|y|) + (\delta_{\rho\delta}\hat{y}_\gamma - \delta_{\gamma\rho}\hat{y}_\delta - \delta_{\gamma\delta}\hat{y}_\rho)\,K_1(m_l|y|)\Big\}, \quad p(|y|) = \frac{2\pi^2}{m_l^2}\ K_0(m_l|y|)$$

$$\hat{y} = y/|y|$$

Lepton loop contribution a_μ^{LbL} in QED

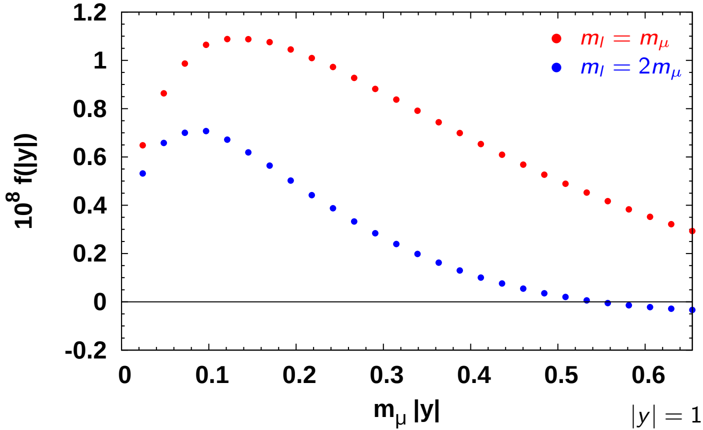
Integrand of lepton loop contribution a_μ^{LbL}:



m_l/m_μ	$a_\mu^{\text{LbL}} \times 10^{11}$ (exact)	$a_\mu^{\text{LbL}} \times 10^{11}$	Precision	Deviation
1/2	1229.07	1257.5(6.2)(2.4)	0.5%	2.3%
1	464.97	470.6(2.3)(2.1)	0.7%	1.2%
2	150.31	150.4(0.7)(1.7)	1.2%	0.06%

1st uncertainty from 3D integration, 2nd uncertainty from extrapolation to small $|y|$.
Behavior for small $|y|$ compatible with $f(|y|) \propto m_\mu |y| \log^2(m_\mu |y|)$.
Analytical results for a_μ^{LbL} with $m_l = m_\mu, 2m_\mu$ reproduced at the percent level.
(Laporta + Remiddi '93, numbers courtesy of Massimo Passera)

Integrand of lepton loop contribution $a_\mu^{\rm LbL}$ (zoom)



$|y| = 1.1$ fm

Behavior for small $|y|$ numerically compatible with $f(|y|) \propto m_\mu |y| \log^2(m_\mu |y|)$.

Conclusions and Outlook

- Explicit master formula for a_μ^{HLbL} in position-space.
- QED kernel function (in continuum and infinite volume to avoid power-law effects $1/L^2$) multiplying the position-space four-point correlation function.
  $\rightarrow$ 6 weight functions stored on disk, ready to be used.
- Verified QED kernel function by evaluating pion-pole contribution in VMD model and lepton-loop in QED (agreement with known results at the percent level).
- Need rather large lattices $L \sim 10\ \mathrm{fm}$ to reproduce results for $m_\pi = 135$ MeV and for $m_{\mathrm{loop}} = m_\mu/2$.
- Study of discretization and finite volume effects: see talk by Nils Asmussen.
- (Near) future: Calculate the four-point correlation function in QCD on the lattice and the HLbL contribution a_μ^{HLbL}.

Invitation to Mainz at the Rhine river

You are cordially invited to the

2nd Workshop of the Muon g-2 Theory Initiative

June 18-22, 2018

Johannes Gutenberg University Mainz, Germany

https://wwwth.kph.uni-mainz.de/g-2/

Local Organizing Committee:

Achim Denig, Georg von Hippel, Harvey Meyer, Andreas Nyffeler, Marc Vanderhaeghen, Hartmut Wittig (Chair)

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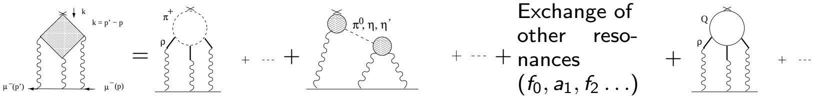
Backup slides

HLbL in muon $g-2$: summary of selected results (model calculations)



de Rafael '94:

Chiral counting:		p^4	p^6	p^8	p^8
N_C-counting:		1	N_C	N_C	N_C

Contribution to $a_\mu \times 10^{11}$:

BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
PdRV:	+105 (26)	-19 (19)	+114 (13)	+8 (12) [f_0, a_1]	+2.3 [c-quark]
N,JN:	+116 (39)	-19 (13)	+99 (16)	+15 (7) [f_0, a_1]	+21 (3)
		ud.: -45	ud.: $+\infty$		ud.: +60

ud. = undressed, i.e. point vertices without form factors

Pseudoscalars: numerically dominant contribution (according to most models !).

Recall (in units of 10^{-11}): $\delta a_\mu(\text{HVP}) \approx 40$; $\delta a_\mu(\text{exp [BNL]}) = 63$; $\delta a_\mu(\text{future exp}) = 16$

BPP = Bijnens, Pallante, Prades '96, '02; HKS = Hayakawa, Kinoshita, Sanda '96, '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts; PdRV = Prades, de Rafael, Vainshtein '09 (compilation; "Glasgow consensus"); N,JN = AN '09; Jegerlehner, AN '09 (compilation)

Recent reevaluations of axial vector contribution lead to much smaller estimates than in MV '04: $a_\mu^{\text{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). Would shift central values of compilations downwards:

$a_\mu^{\text{HLbL}} = (98 \pm 26) \times 10^{-11}$ (PdRV) and $a_\mu^{\text{HLbL}} = (102 \pm 39) \times 10^{-11}$ (N, JN).

HLbL in muon $g-2$: summary of selected results for $a_\mu^{\mathrm{HLbL}} \times 10^{11}$

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

- Pseudoscalars dominate numerically. Other contributions not negligible. Cancellation between π, K-loops and quark loops !
- Recent reevaluations of axial vector contribution lead to much smaller estimates than in MV '04: $a_\mu^{\mathrm{HLbL;axial}} = (8 \pm 3) \times 10^{-11}$ (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15). This would shift central values of compilations downwards: $a_\mu^{\mathrm{HLbL}} = (98 \pm 26) \times 10^{-11}$ (PdRV) and $a_\mu^{\mathrm{HLbL}} = (102 \pm 39) \times 10^{-11}$ (N, JN).
- PdRV: Analyzed results obtained by different groups with various models and suggested new estimates for some contributions (shifted central values, enlarged errors). Do not consider dressed light quark loops as separate contribution. Added all errors in quadrature !
- N, JN: New evaluation of pseudoscalar exchange contribution imposing new short-distance constraint on off-shell form factors. Took over most values from BPP, except axial vectors from MV. Added all errors linearly.

Model calculations of HLbL: recent developments

- Most calculations for neutral pion and all light pseudoscalars agree at level of 15%, but full range of estimates (central values) much larger:

$$a_\mu^{\mathrm{HLbL};\pi^0} = (50-80)\times 10^{-11} = (65\pm 15)\times 10^{-11} \quad (\pm 23\%)$$
$$a_\mu^{\mathrm{HLbL;P}} = (59-114)\times 10^{-11} = (87\pm 27)\times 10^{-11} \quad (\pm 31\%)$$

- New estimates for axial vectors (Pauk, Vanderhaeghen '14; Jegerlehner '14, '15):

$$a_\mu^{\mathrm{HLbL;axial}} = (6-8)\times 10^{-11}$$

  Substantially smaller than in MV '04 !

- First estimate for tensor mesons (Pauk, Vanderhaeghen '14):

$$a_\mu^{\mathrm{HLbL;tensor}} = 1\times 10^{-11}$$

- Open problem: Dressed pion-loop
  Potentially important effect from pion polarizability and a_1 resonance
  (Engel, Patel, Ramsey-Musolf '12; Engel '13; Engel, Ramsey-Musolf '13):

$$a_\mu^{\mathrm{HLbL};\pi-\mathrm{loop}} = -(11-71)\times 10^{-11}$$

  Maybe large negative contribution, in contrast to BPP '96, HKS '96.
  Not confirmed by recent reanalysis by Bijnens, Relefors '15, '16. Essentially get again old central value from BPP, but smaller error estimate:

$$a_\mu^{\mathrm{HLbL};\pi-\mathrm{loop}} = (-20\pm 5)\times 10^{-11}$$

- Open problem: Dressed quark-loop
  Dyson-Schwinger equation approach (Fischer, Goecke, Williams '11, '13):

$$a_\mu^{\mathrm{HLbL;quark-loop}} = 107\times 10^{-11} \qquad \text{(still incomplete !)}$$

  Large contribution, no damping seen, in contrast to BPP '96, HKS '96.