

# Pion Transition Form Factor from Lattice QCD in Position Space

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## Matrix Element

### Neutral Pion Transition Form Factor

$$\begin{aligned}
 M_{\mu\nu}(q_1, q_2) &= \int d^4u e^{-iq_1 \cdot u - iq_2 \cdot v} \langle 0 | T \{ iJ_\mu(u) iJ_\nu(v) \} | \pi^0(p) \rangle \\
 &= \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} q_{1,\rho} q_{2,\sigma} F(q_1^2, q_2^2)
 \end{aligned}$$

### Coordinate Space Formulation

$$\langle 0 | T \{ iJ_\mu(u) iJ_\nu(v) \} | \pi^0(p) \rangle = \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} (-i\partial_\rho^u) (-i\partial_\sigma^v) \left[ F'(\mathbf{p} \cdot (u-v), (u-v)^2) e^{ipv} \right]$$

Give the Result with the Pion Operator:

$$= \langle 0 | \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} \int_0^1 dx \left[ -\partial_\rho^u F_c(x, (u-v)^2) \right] \partial_\sigma \pi^0(xu + (1-x)v) | \pi^0(\vec{p}) \rangle$$

where

$$F'(\mathbf{p} \cdot (u-v), (u-v)^2) = \int_{-\infty}^{\infty} dx F_c(x, (u-v)^2) e^{ixp \cdot (u-v)}$$

$$F_c(x, (u-v)^2) = 0 \text{ if } x < 0 \text{ or } x > 1$$

## Proof

When  $u_t = v_t$  (also  $p \cdot (u - v)$  is a real number), define  $F_c(x, (u - v)^2)$  to be the Fourier transformation of  $F'(p \cdot (u - v), (u - v)^2)$ :

$$F'(p \cdot (u - v), (u - v)^2) = \int_{-\infty}^{\infty} dx F_c(x, (u - v)^2) e^{ixp \cdot (u - v)}$$

Then, we have:

$$\langle 0 | T \{ iJ_\mu(u) iJ_\nu(v) \} | \pi^0(p) \rangle = \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} (-i\partial_\rho^u) (-i\partial_\sigma^v) \int_{-\infty}^{\infty} dx F_c(x, (u - v)^2) e^{ip \cdot (xu + (1-x)v)}$$

Consider the following three point function (assuming  $u_t, v_t > w_t$ ) by inserting the pion projection operator  $\hat{P}_{\pi^0} = \int \frac{d^3\vec{p}}{(2\pi)^3} | \pi^0(\vec{p}) \rangle \frac{1}{2E_{\pi^0, \vec{p}}} \langle \pi^0(\vec{p}) |$ :

$$\begin{aligned} & \langle 0 | T \{ iJ_\mu(u) iJ_\nu(v) \} \hat{P}_{\pi^0} \pi^0(w) | 0 \rangle \\ &= \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} \int_{-\infty}^{\infty} dx [-\partial_\rho^u F_c(x, (u - v)^2)] \partial_\sigma G(xu + (1-x)v - w) \end{aligned}$$

$F_c(x, (u - v)^2)$  in  $x \in [0, 1]$ 

Select  $\omega = x'u + (1 - x')v + \varepsilon$ , where  $\varepsilon$  is a very small distance, we get:

$$\begin{aligned} & \langle 0 | T \{ iJ_\mu(u) iJ_\nu(v) \} \hat{P}_{\pi^0} \pi^0(\omega) | 0 \rangle \\ &= \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} \int_{-\infty}^{\infty} dx [-\partial_\rho^u F_c(x, (u - v)^2)] \partial_\sigma G((x - x')(u - v) - \varepsilon) \end{aligned}$$

In the area of  $x' > 1$  or  $x' < 0$ ,  $\langle 0 | T \{ iJ_\mu(u) iJ_\nu(v) \} \hat{P}_{\pi^0} \pi^0(\omega) | 0 \rangle$  should be finite when  $\varepsilon \rightarrow 0$ , but the above formula suggest a singularity behavior of  $\partial_\sigma G((x - x')(u - v) - \varepsilon)$ . This implies that:

$$F_c(x, (u - v)^2) = 0 \text{ if } x < 0 \text{ or } x > 1$$

Then the integral becomes from 0 to 1:

$$\begin{aligned} & \langle 0 | T \{ iJ_\mu(u) iJ_\nu(v) \} | \pi^0(\vec{p}) \rangle \\ &= \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} \int_0^1 dx [-\partial_\rho^u F_c(x, (u - v)^2)] i p_\sigma e^{ip \cdot (xu + (1-x)v)} \end{aligned}$$

or, in pion operator expression:

$$= \langle 0 | \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} \int_0^1 dx [-\partial_\rho^u F_c(x, (u - v)^2)] \partial_\sigma \pi^0(xu + (1 - x)v) | \pi^0(\vec{p}) \rangle$$

# Convert $F_c$ to Momentum Space Form Factor and the Property of $F_c$

Let  $v = 0$  and  $p = q_1 + q_2$ , we get mapping from  $F_c$  to momentum space form factor:

$$F(q_1^2, q_2^2) = \int d^4 u e^{-iq_1 \cdot u} \int_0^1 dx F_c(x, u^2) e^{ixp \cdot u}$$

In the Chiral limit and  $\vec{p} = 0$ ,  $q_1 = 0$ ,  $F(q_1^2 \rightarrow 0, q_2^2 \rightarrow 0) = 1$ , and:

$$\int d^4 u \int_0^1 dx F_c(x, u^2) = 1$$

## OPE analysis

When  $u$  and  $v$  are very close, we have such approximation [Gerardin et al., 2016]:

$$T\psi(u)\bar{\psi}(v) \approx \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip \cdot (u-v)}}{ip_\rho \gamma_\rho + m} = \frac{(u-v)_\rho \gamma_\rho}{2\pi^2((u-v)^2)^2}$$

$$\begin{aligned} & T [i\bar{\psi}(u)\gamma_\mu\psi(u)] [i\bar{\psi}(v)\gamma_\mu\psi(v)] \\ & \approx -\frac{(u-v)_\rho}{2\pi^2((u-v)^2)^2} \bar{\psi}(u)\gamma_\mu\gamma_\rho\gamma_\nu\psi(v) - \frac{(v-u)_\rho}{2\pi^2((v-u)^2)^2} \bar{\psi}(v)\gamma_\nu\gamma_\rho\gamma_\mu\psi(u) \\ & = -\frac{\epsilon_{\mu\nu\rho\sigma}(u-v)_\rho}{2\pi^2((u-v)^2)^2} [\bar{\psi}(u)\gamma_\sigma\gamma_5\psi(v) + \bar{\psi}(v)\gamma_\sigma\gamma_5\psi(u)] \end{aligned}$$

Consider the definition of pion form factor:

$$\langle 0 | \bar{u}(u)\gamma_\sigma\gamma_5 u(u) | \pi^0(\vec{p}) \rangle = F_\pi p_\sigma e^{ip \cdot u} = \langle 0 | -iF_\pi \partial_\sigma \pi^0(u) | \pi^0(\vec{p}) \rangle$$

Analogously, the two current operator can be shown as follow:

$$T\{iJ_\mu(u)iJ_\nu(v)\} \xrightarrow{\mu, \nu \text{ are close}} \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} 2(u-v)_\rho \left[ \frac{F_\pi^2}{3} \frac{1}{((u-v)^2)^2} \right] \partial_\sigma \pi^0\left(\frac{u+v}{2}\right)$$

# Parametrization

Do Fourier series expansion of  $F_c(x, (u - v)^2)$ , using the fact that  $0 \leq x \leq 1$ :

$$\begin{aligned}
 -\partial_\rho^u F_c(x, (u - v)^2) &= 2(u - v)_\rho \left[ \frac{F_\pi^2}{3} \frac{1}{((u - v)^2)^2} \right] \\
 &\quad \times \sum_{n=0}^{\infty} f_n(|u - v|) \frac{(2n + 1)\pi}{2} \sin((2n + 1)\pi x)
 \end{aligned}$$

$$\begin{aligned}
 &\langle 0 | T i J_\mu(u) i J_\nu(v) | \pi^0(\vec{p}) \rangle \\
 &= \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} 2(u - v)_\rho i p_\sigma \left[ \frac{F_\pi^2}{3} \frac{1}{((u - v)^2)^2} \right] \\
 &\quad \times \sum_{n=0}^{\infty} f_n(|u - v|) \frac{(2n + 1)\pi}{2} \int_0^1 \sin((2n + 1)\pi x) e^{i p \cdot (xu + (1-x)v)} dx
 \end{aligned}$$

## Study $r$ Dependence of the Coordinate Space Formulation

Define  $f(|r|)$ :

$$\int_0^1 dx [-\partial_\rho^u F_c(x, r^2)] = 2r_\rho \left[ \frac{F_\pi^2}{3} \frac{1}{(r^2)^2} \right] f(|r|)$$

that is:

$$f(|r|) = \sum_{n=0}^{\infty} f_n(|r|)$$

Let  $u = r/2$ ,  $v = -r/2$  and  $r_t = 0$ ,  $\vec{p} = 0$  to reserve the factor  $f_n(|r|)$  in Fourier series, we have:

$$\langle 0 | T iJ_\mu(0, \vec{r}/2) iJ_\nu(0, -\vec{r}/2) | \pi^0(\vec{p} = 0) \rangle = \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} 2r_\rho i p_\sigma \left[ \frac{F_\pi^2}{3} \frac{1}{(r^2)^2} \right] f(|r|)$$

Remember we also have the constrains from the pion width, which imply  $f_n|r|$  should satisfy:

$$\frac{\pi^2}{2} \int_0^\infty \frac{F_\pi^2}{3} f(|r|) 2r dr = 1$$

# Point Source Propagator and Contraction

Propagator:

- 1024 point source propagator in each configuration
- 256 random area group chosen from the lattice
- 4 random points per group

Three-Point Correlation Function:

$$C_{\mu\nu}^{\text{conn}}(|\vec{r}|, \vec{p}, t_\pi) = \sum_{|\vec{r}'|=|\vec{r}|, \vec{x}} \langle 0 | J_\mu(0, \vec{r}'/2) J_\nu(0, -\vec{r}'/2) P^+(\vec{x}, t_\pi) | 0 \rangle e^{i\vec{p}\cdot\vec{x}}$$

# Ensembles

- 24c64 Ensemble

Observable	Fit	% err.
$am'_{\text{res}}(m_l)$	0.0022824(70)	0.31
$am_\pi$	0.13975(10)	0.07
$am_K$	0.504154(89)	0.02
$am_\Omega$	1.6726(25)	0.15
$am'_\Omega$	2.040(63)	3.09
$af_\pi$	0.13055(11)	0.09
$af_K$	0.15815(13)	0.09
$Z_A$	0.73457(11)	0.02
$Z_V^\pi$	0.72672(35)	0.05
$Z_V^K$	0.7390(11)	0.15
$aE_{\pi\pi}^2$	0.28175(21)	0.08
$a\delta E_\pi^2$	0.002246(52)	2.31
$ap$	0.01775(21)	1.16
$\delta_0^2(p)$	-0.339(12)°	3.40
$m_\pi a_0^2$	-0.0464(10)	2.23

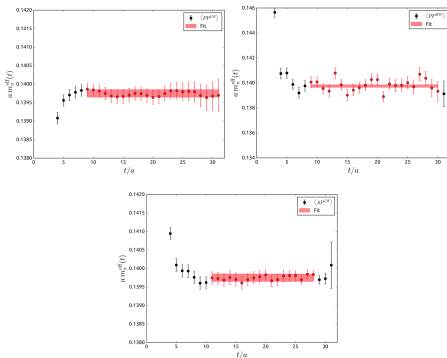


Figure: pion mass

## Ensembles

## ● 48c96 Ensemble

Observable	Fit	% err.
$am'_{\text{res}}(m_l)$	0.0006102(32)	0.53
$am_\pi$	0.08048(13)	0.16
$am_K$	0.28856(14)	0.05
$am_\Omega$	0.97019(64)	0.07
$am'_\Omega$	1.2724(50)	0.39
$af_\pi$	0.07592(11)	0.14
$af_K$	0.090562(85)	0.09
$Z_A$	0.711893(73)	0.01
$Z_V^\pi$	0.71060(36)	0.05
$Z_V^K$	0.71464(57)	0.08
$aE_{\pi\pi}^2$	0.16165(30)	0.18
$a\delta E_\pi^2$	0.00068(19)	28.11
$ap$	0.0074(10)	13.97
$\delta_0^0(p)$	-0.201(84) <sup>o</sup>	41.83
$m_\pi a_0^2$	-0.038(10)	27.26

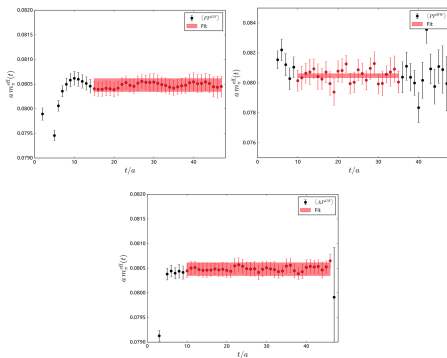
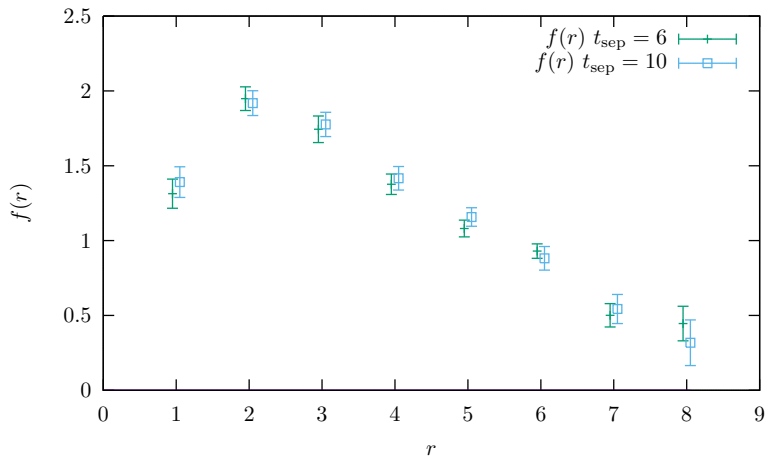
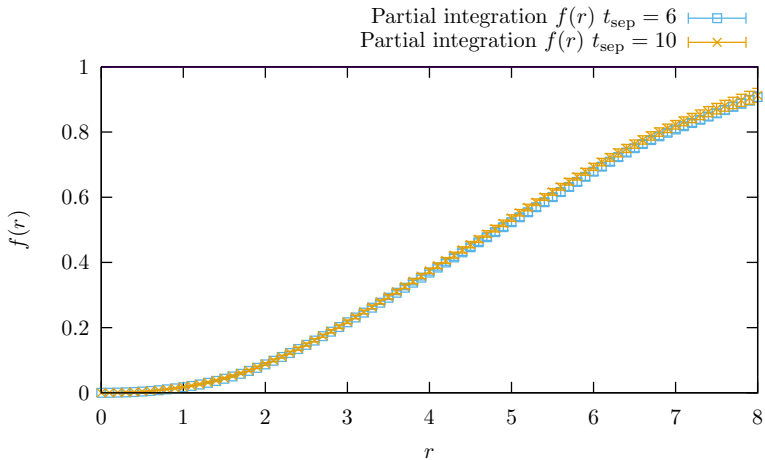


Figure: pion mass

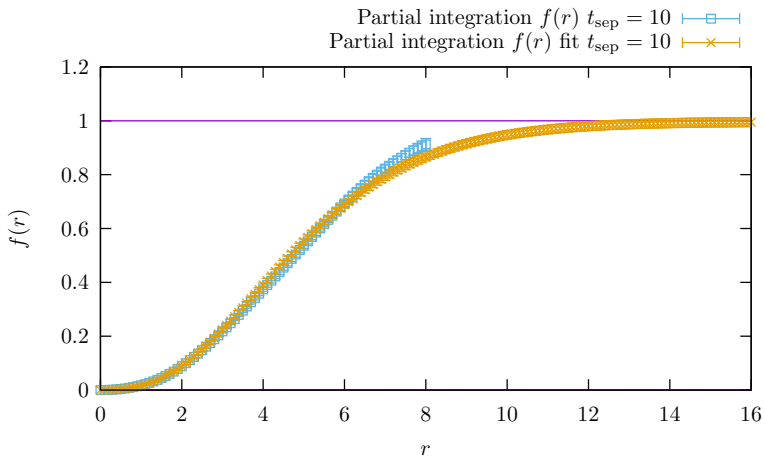
## Plots

24c64 Lattice, 1024 Point Source Propagator









Integral result:

$$\frac{\pi^2}{2} \int_0^\infty \frac{F_\pi^2}{3} f_{\text{fit}}(r) 2r dr = 0.9965 \pm 0.017$$

# Future Plan

- Compute in 48c96 ensemble
- Compute  $n$  dependent  $f_n(|r|)$  (not restrict to  $r_t = 0$  or  $\vec{p} = 0$ )