The pion transition form factor from lattice QCD

Antoine Gérardin

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

In collaboration with Harvey Meyer and Andreas Nyffeler

Based on [Phys. Rev. D **94**, 074507 (2016)]

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

PRISMA

March 14, 2018

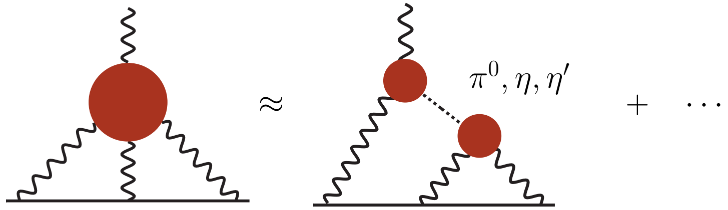
Muon g-2 Theory Initiative Hadronic Light-by-Light working group workshop

Motivations

- Pion-pole contribution to the hadronic light-by-light scattering contribution to the $(g-2)_\mu$

  → dominant contribution in model calculations

  → dispersive approach [Colangelo et al. '14, '15]



Frequent estimates :

$a_\mu^{\rm HLbL}(\pi^0) \approx 67.0 \times 10^{-11}$

$a_\mu^{\rm HLbL}(\eta) \approx 14.5 \times 10^{-11}$

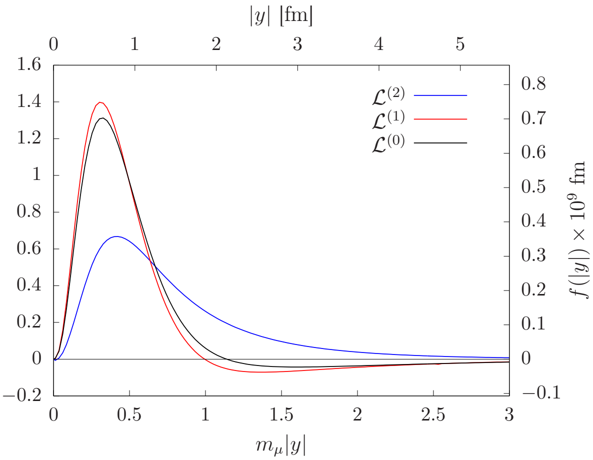
$a_\mu^{\rm HLbL}(\eta') \approx 12.5 \times 10^{-11}$

- Estimation of the finite-size effects of the full lattice QCD calculation

  → the pion-pole contribution is long-range

  → compute the transition form factor on the same ensembles

  → use the result to remove the dominant FSE



- Tests of the dynamics of QCD

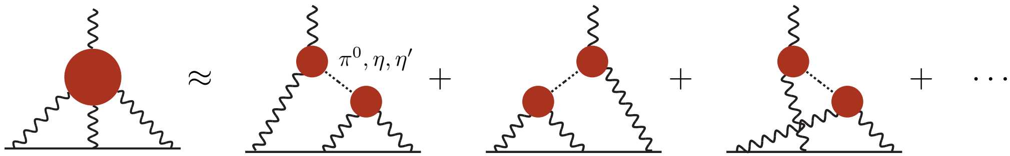
  → Chiral anomaly

  → Short-distance constraints : Brodsky-Lepage behaviour, OPE prediction

Motivations : the pion-pole contribution

[Jegerlehner & Nyffeler '09]

$\tau = \cos(\theta)$
$Q_1 \cdot Q_2 = Q_1 Q_2 \cos(\theta)$



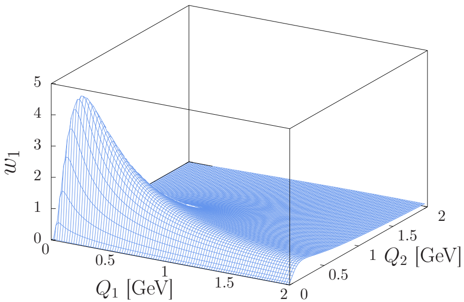
$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) \; + \\ w_2(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

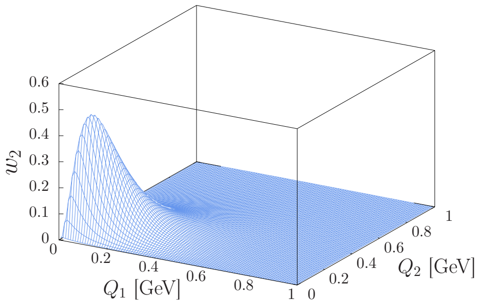
→ Product of one single-virtual and one double-virtual transition form factors

→ $w_{1,2}(Q_1, Q_2, \tau)$ are model-independent weight functions

→ Spacelike virtualities

→ Weight functions are concentrated at small momenta below 1 GeV (here for $\tau = -0.5$)



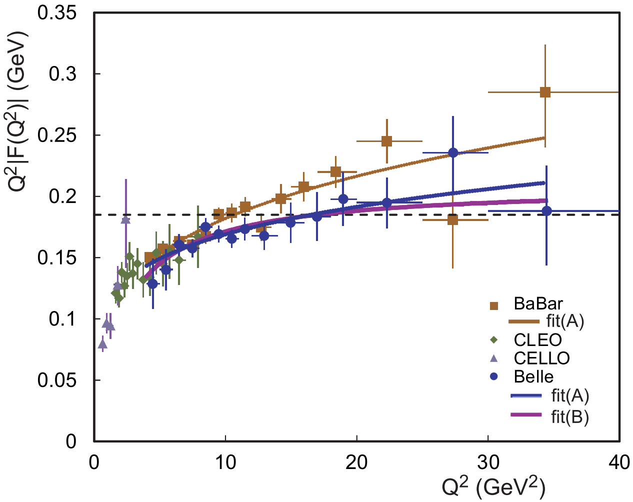


↪ Need the pion TFF for arbitrary space-like virtualities in the momentum range $[0-3]$ GeV^2

The pion transition form factor

- The single-virtual transition form factor has been measured (CELLO, CLEO, BaBar, Belle)
  - $\rightarrow$ Belle data agree with Brodsky-Lepage ($\sim 1/Q^2$)
  - $\rightarrow$ Belle and Babar results are quite different
  - $\rightarrow$ No measurement for $Q < 0.8$ GeV (dominant contribution)
  - $\rightarrow$ New results planed at BESIII

[Belle '12]



- No result yet for the double-virtual transition form factor
  - $\hookrightarrow$ measurement planned at BESIII
  - $\hookrightarrow$ challenging (small cross section)

- Adler-Bell-Jackiw (ABJ) anomaly in the chiral limit : $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \dfrac{1}{4\pi^2 F_\pi}$

- Short-distance constraints

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,0) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{Q^2}$$

Brodsky-Lepage behavior

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3Q^2}$$

OPE prediction

Lattice calculation

Lattice calculation

In Minkowski space-time :

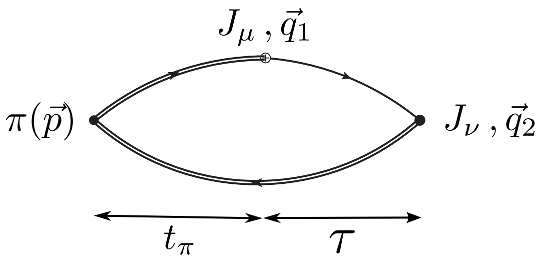
$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = i \int \mathrm{d}^4x\, e^{iq_1 x}\, \langle 0 | T\{J_\mu(x) J_\nu(0)\} | \pi^0(p) \rangle = M_{\mu\nu}(q_1^2, q_2^2)$$

- $J_\mu(x)$ hadronic component of the electromagnetic current : $J_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \ldots$

In Euclidean space-time : [Ji & Jung '01] [Cohen et al. '08] [Feng et al. '12]

$$M_{\mu\nu}^E(q_1^2, q_2^2) = -\int \mathrm{d}\tau\, e^{\omega_1 \tau} \int \mathrm{d}^3z\, e^{-i\vec{q}_1 \vec{z}}\, \langle 0 | T\left\{ J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) \right\} | \pi(p) \rangle$$

- Analytical continuation : $q_1 = (\omega_1, \vec{q}_1)$
- We must kept $q_{1,2}^2 < M_V^2 = \min(M_\rho^2, 4m_\pi^2)$ to avoid poles



The main object to compute is the Euclidean three-point correlation function :

$$C_{\mu\nu}^{(3)}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) = \sum_{\vec{x}, \vec{z}} \left\langle T\left\{ J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0) \right\} \right\rangle e^{i\vec{p}\vec{x}}\, e^{-i\vec{q}_1 \vec{z}}$$

- We use one local and one conserved vector current. We focus on $\vec{p} = \vec{0}$ and compute all $\vec{q}_1$

Kinematic reach in the photon virtualities

Photons virtualities (pion rest frame) :

$$q_1^2 = \omega_1{}^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$

$\Rightarrow |\vec{q}_1|^2 = (2\pi/L)^2 |\vec{n}|^2 \quad , \quad |\vec{n}|^2 = 1, 2, 3, 4, 5, \ldots$

$\Rightarrow \omega_1$ is a (real) free parameter

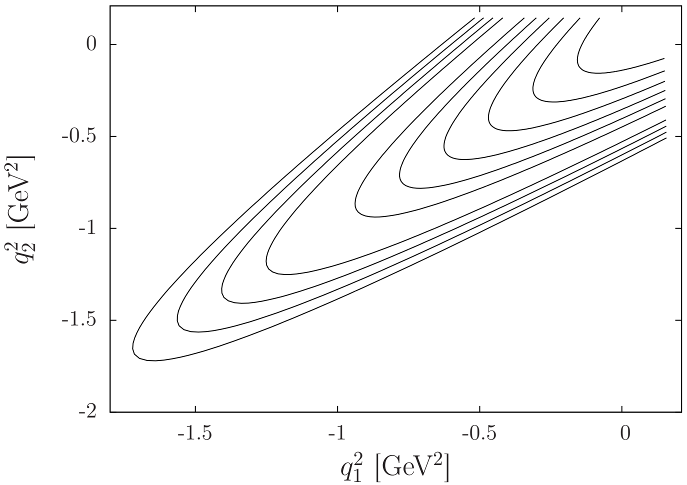


Figure – $L/a = 48$ at $a = 0.065$ fm.

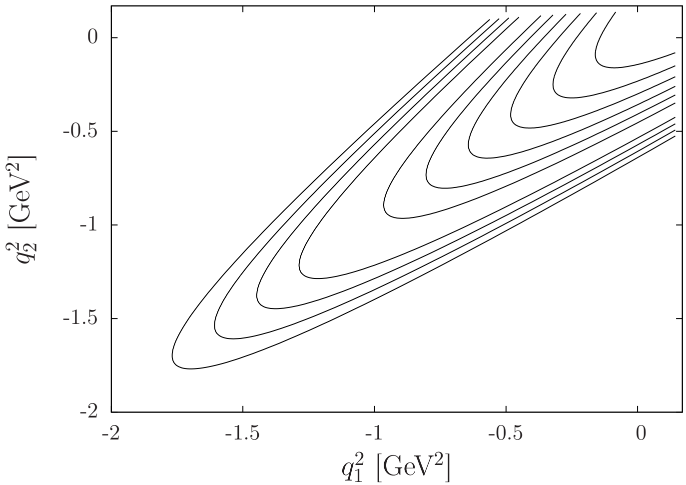


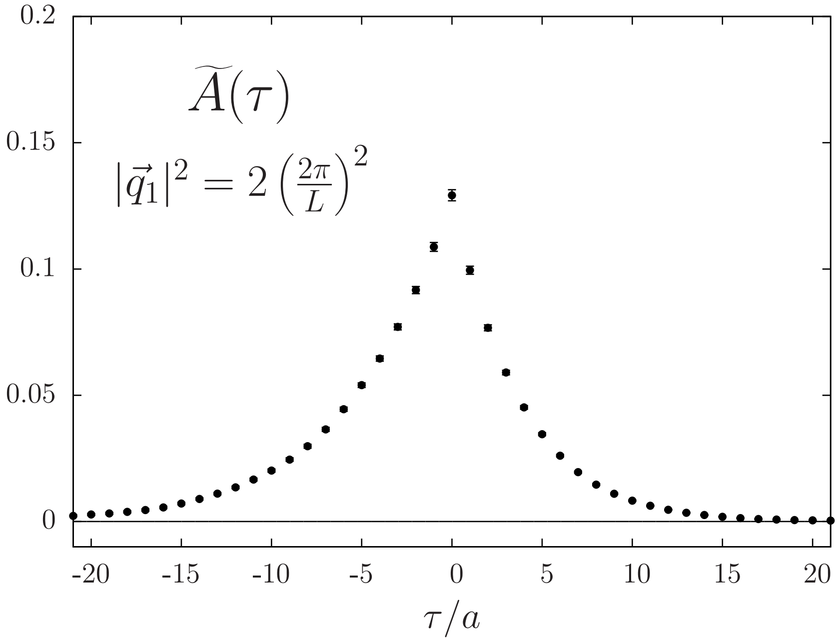
Figure – $L/a = 64$ at $a = 0.048$ fm.

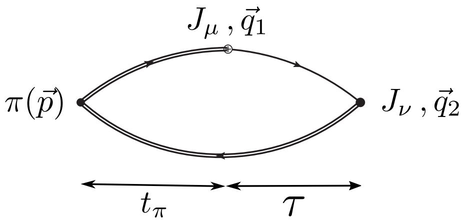
Shape of the integrand for F7 $(a = 0.065\ \mathrm{fm}$ and $m_\pi = 270\ \mathrm{MeV})$

$$\mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \propto \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau\, \widetilde{A}_{\mu\nu}(\tau)\, e^{\omega_1 \tau}$$

$$A_{\mu\nu}(\tau) = \lim_{t_\pi \to \infty} C_{\mu\nu}(\tau, t_\pi) e^{E_\pi t_\pi}$$

$$\widetilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau)\, e^{-E_\pi \tau} & \tau < 0 \end{cases}$$





On the lattice :

- Discrete sum over lattice points :

$$\int \mathrm{d}\tau \to a \sum_\tau$$

- Finite size of the box :

$$|\tau| \leq \tau_{\max} \neq \infty$$

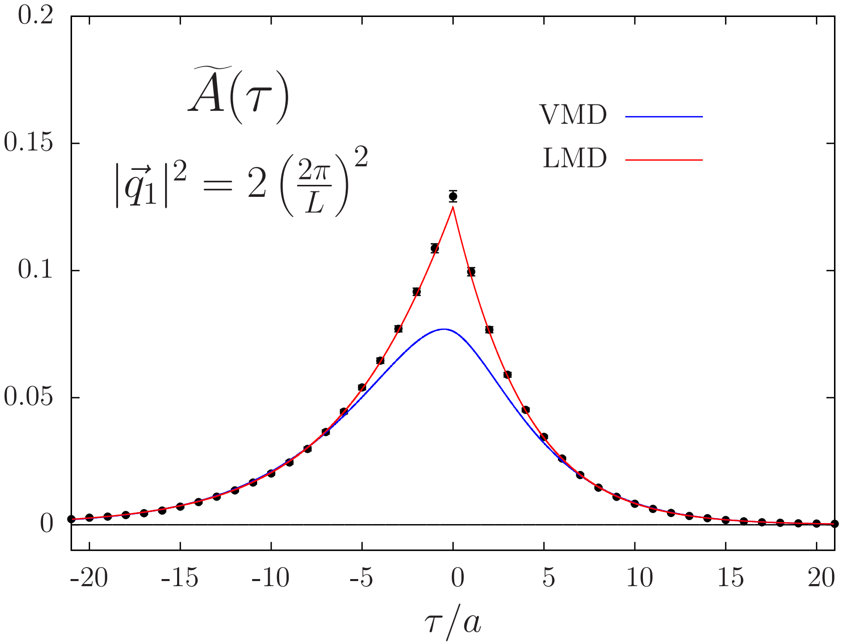
- Signal deteriorates at large $|\tau|$: $e^{\omega_1 \tau}$

Shape of the integrand for F7 ($a = 0.065$ fm and $m_\pi = 270$ MeV)

- The vector meson dominance (VMD) model is expected to give a good description of the data at large τ

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)} \quad \rightarrow \quad \widetilde{A}(\tau) = \ \dots \quad \text{(known analytical expression)}$$

- Fit the data at large τ and use the result of the fit for $\tau > \tau_c \gtrsim 1.3$ fm



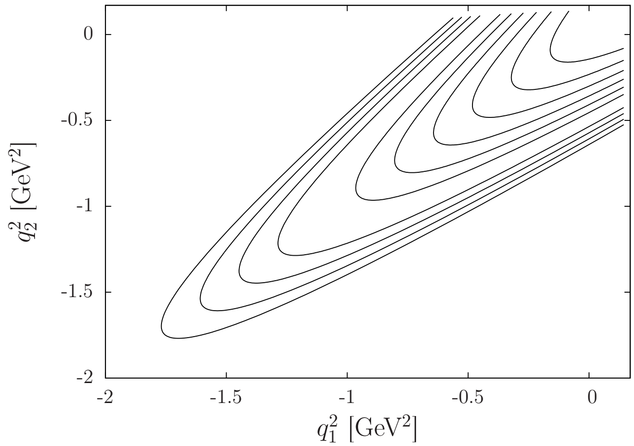
- Check the dependance on the model using LMD rather than VMD :

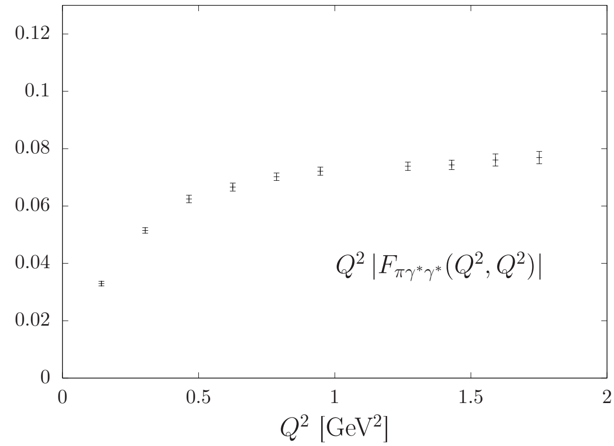
$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

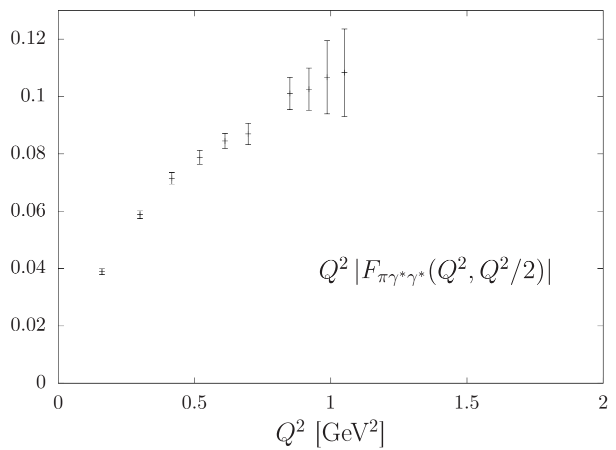
- The difference between the two models is included in the systematic error.

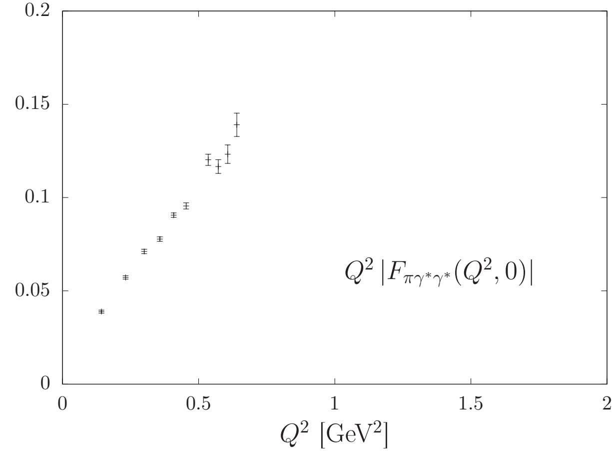
Results with $N_f = 2$

Results with $N_f = 2$ (for one ensemble with $m_\pi = 270$ MeV and $a = 0.05$ fm)









Results with $N_f = 2$

- In our published paper [Phys. Rev. D **94**, 074507 (2016)] : $N_f = 2$ Wilson-Clover ensembles (CLS)

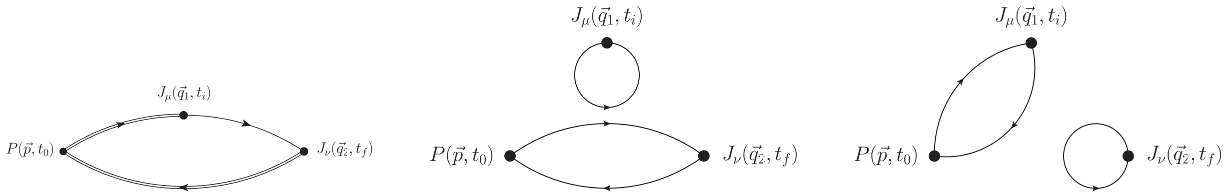
  $\hookrightarrow$ Several pion masses (down to 190 MeV) and several lattice spacing (down to 0.048 fm)

  $\hookrightarrow$ Continuum and chiral limits

  $\hookrightarrow$ Quark connected and disconnected contributions

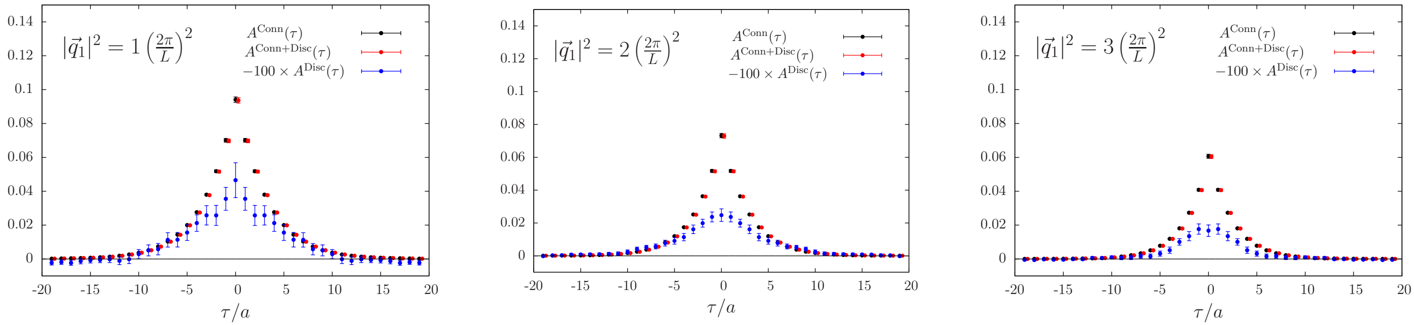
Results with $N_f = 2$

- In our published paper [Phys. Rev. D **94**, 074507 (2016)] : $N_f = 2$ Wilson-Clover ensembles (CLS)
  - ↪ Several pion masses (down to 190 MeV) and several lattice spacing (down to 0.048 fm)
  - ↪ Continuum and chiral limits
  - ↪ Quark connected and disconnected contributions



→ Disconnected contributions are known to be much more challenging on the lattice

→ Computed on one ensemble only ($a = 0.065$ fm, $m_\pi = 440$ MeV)



→ The disconnected contribution is below 1% (but could be enhanced at smaller pion masses ...)

Results with $N_f = 2$

- In our published paper [Phys. Rev. D **94**, 074507 (2016)] : $N_f = 2$ Wilson-Clover ensembles (CLS)

  ↪ Several pion masses (down to 190 MeV) and several lattice spacing (down to 0.048 fm)

  ↪ Continuum and chiral limits

  ↪ Quark connected and disconnected contributions

- We used phenomenological models to describe our data and extrapolate to the physical points

- VMD model

$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- LMD model (Lowest Meson Dominance) [Moussallam '94] [Knecht et al. '99]

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- LMD+V model [Knecht & Nyffeler '01]

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2 + q_2^2) + \widetilde{h}_1 (q_1^2 + q_2^2)^2 + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2 + q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

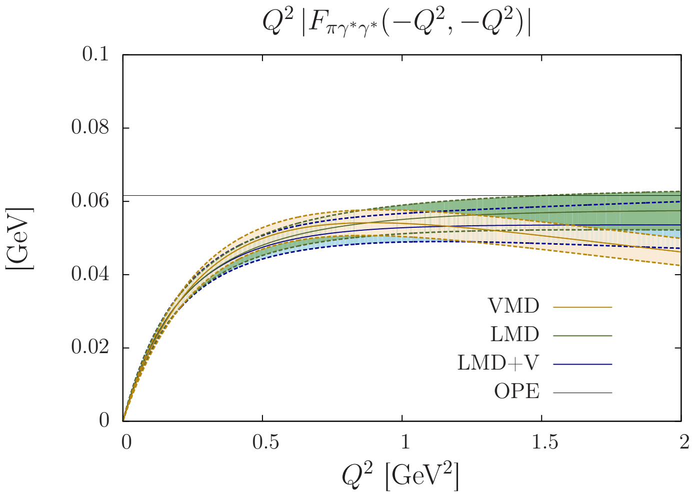
↪ All fit parameters were not fitted (M_{V_2} ...)

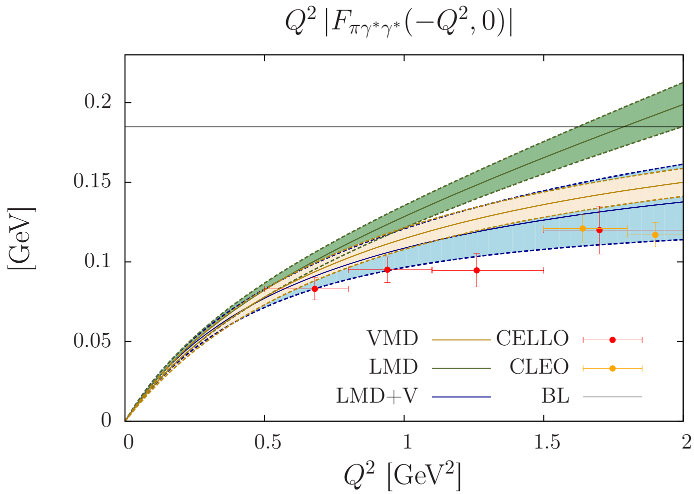
- We assumed a simple dependance on the lattice spacing and pion mass

$$\alpha(m_\pi, a) = \alpha(m_\pi, a) + \gamma_a\, a + \gamma_{m_\pi}\, m_\pi \;\; \rightarrow$$ global fit of the eight lattice ensembles

Final results for the form factor (at the physical point)

- Results in the continuum and chiral limit for each model





- In practice we know the TFF for all Q_1^2 and Q_2^2, not only $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)$ and $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$
- VMD model : bad χ^2 $\rightarrow$ wrong asymptotic behavior in the double-virtual case
- Results for LMD and LMD+V have good χ^2 and are in good agreement with phenomenology :
  $\alpha^{\mathrm{LMD}} = 0.275(18)(3)\ \mathrm{GeV}^{-1}$ and $\alpha^{\mathrm{LMD+V}} = 0.273(24)(7)\ \mathrm{GeV}^{-1}$ compare well with the theoretical prediction $\alpha^{\mathrm{th}} = 0.274\ \mathrm{GeV}^{-1}$.

Back to phenomenology : the pion-pole contribution

[Jegerlehner & Nyffeler '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)$$

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

→ most model calculations yield results in the range

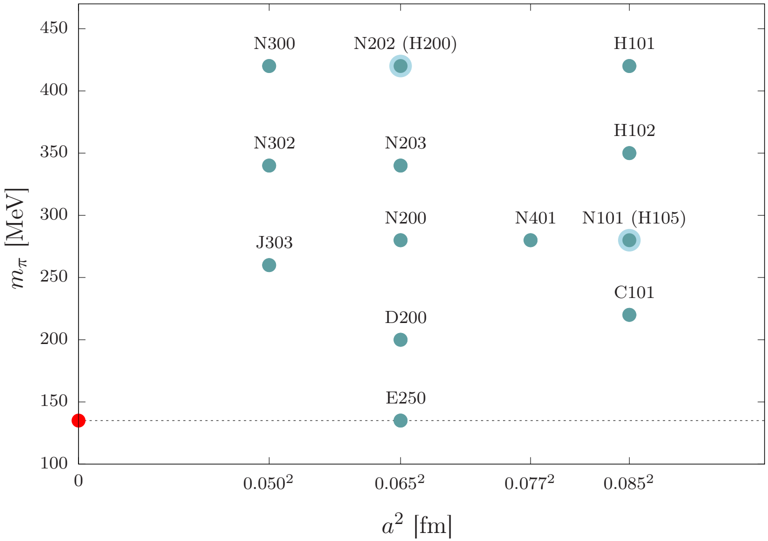
$$a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11}$$

Model	$a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$
LMD (this work)	68.2(7.4)
LMD+V (this work)	65.0(8.3)
VMD (theory)	57.0
LMD (theory)	73.7
LMD+V (theory + phenomenology)	62.9

Λ [GeV]	LMD		LMD+V	
0.25	14.6	(21.4%)	14.4	(22.1%)
0.5	37.9	(55.5%)	37.2	(57.2%)
0.75	50.7	(74.4%)	49.5	(76.1%)
1.0	57.3	(84.0%)	55.5	(85.4%)
1.5	62.9	(92.3%)	60.6	(93.1%)
2.0	65.1	(95.5%)	62.5	(96.1%)
5.0	67.7	(99.2%)	64.6	(99.4%)
20.0	68.2	(100%)	65.0	(100%)

Preliminary results with $N_f = 2 + 1$

The pion transition form factor with $N_f = 2+1$

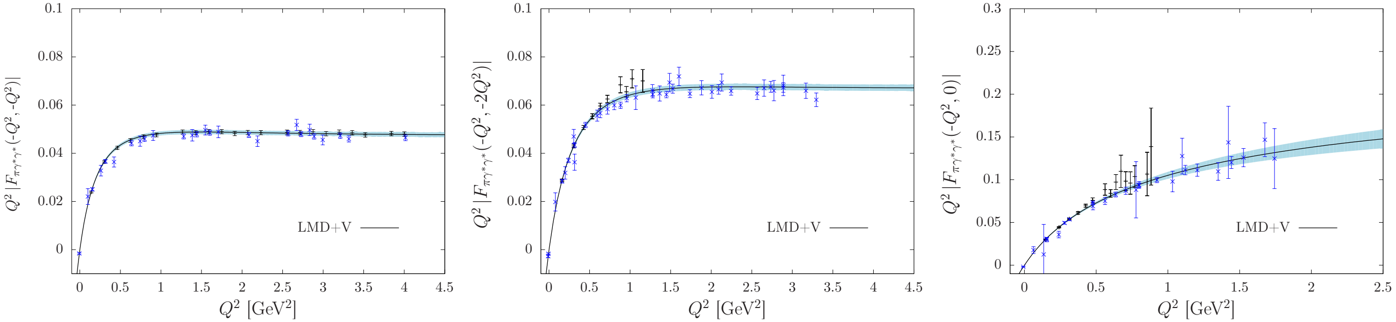


- $N_f = 2+1$: dynamical charm quark
- Add improvement of the vector current
  - ↪ Full $\mathcal{O}(a)$-improvement
  - ↪ Smaller discretization effects
  - ↪ local-local and local-conserved
- Ensembles with different volumes
  - ↪ can directly check finite-size effects
- Chiral and continuum extrapolations under control

- Add a new kinematic with $\vec{p} \neq \vec{0}$ (in addition to the pion rest frame)
  - ↪ access to many more points in the (Q_1^2, Q_2^2) plane
- Compute all spatial momenta $\vec{q}_1$ to cover the plane $0 < Q_1^2, Q_2^2 < 3~\mathrm{GeV}^2$
  - ↪ Important to reduce statistical noise (multiplicity)

Preliminary results with $N_f = 2+1$

- Ensemble $m_\pi \approx 280$ MeV at $a \approx 0.065$ fm
- Color code :
  - Black points : pion rest frame
  - Blue points : pion with on unit of momenta $2\pi/L$



- good statistical precision
- data with $\vec{p} \neq \vec{0}$ are valuable !

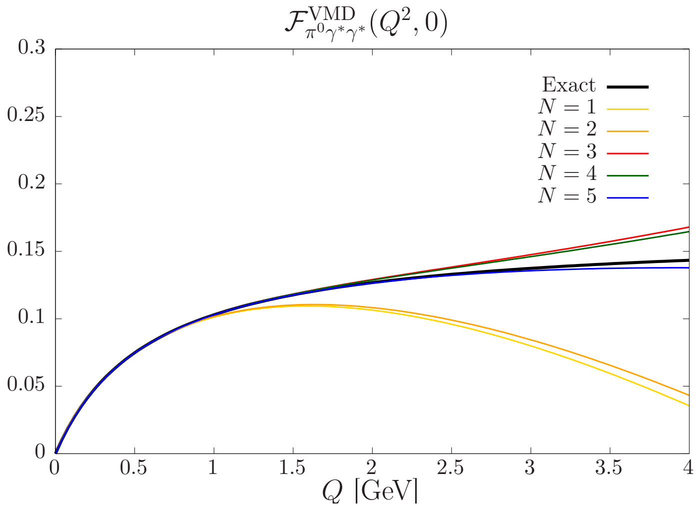
  $\hookrightarrow$ but require one more (sequential) inversion of the Dirac operator
- We have now data on all 14 ensembles (physical pion mass still missing), analysis ongoing.
- We are trying several methods to obtain a model independent determination of $a_\mu^{\mathrm{HLbL};\pi^0}$

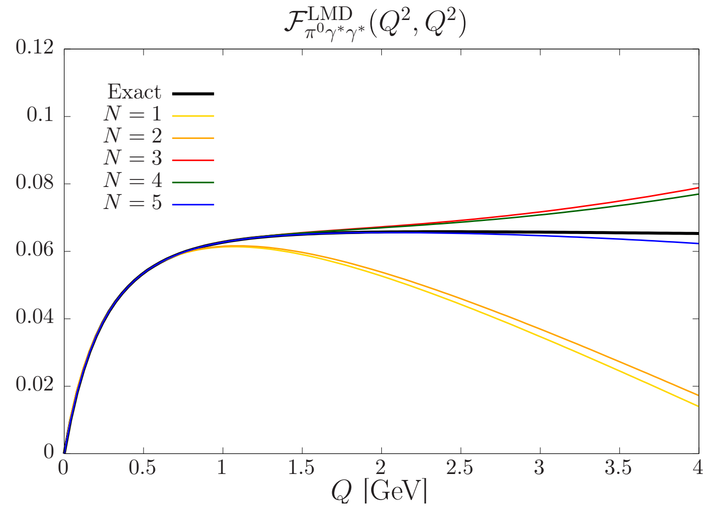
z-expansion inspired fits

- Assumes the following double z-expansion for spacelike momenta : $F(Q_1^2, Q_2^2) = \sum_{n,m=0}^{N} c_{nm}\, z_1^n\, z_2^n$

where $c_{nm} = c_{mn}$ and with
$$z_i = \frac{\sqrt{t_c + Q_i^2} - \sqrt{t_c - t_0}}{\sqrt{t_c + Q_i^2} + \sqrt{t_c - t_0}}\,, \quad t_0 = t_c\left(1 - \sqrt{1 + Q^2_{\mathrm{max}}/t_c}\right)$$

- t_0 is the optimal choice which reduces the maximum value of $|z_i|$ in the range $[0, Q^2_{\mathrm{max}}]$
- For the VMD or LMD models, one can compute the coefficients exactly :





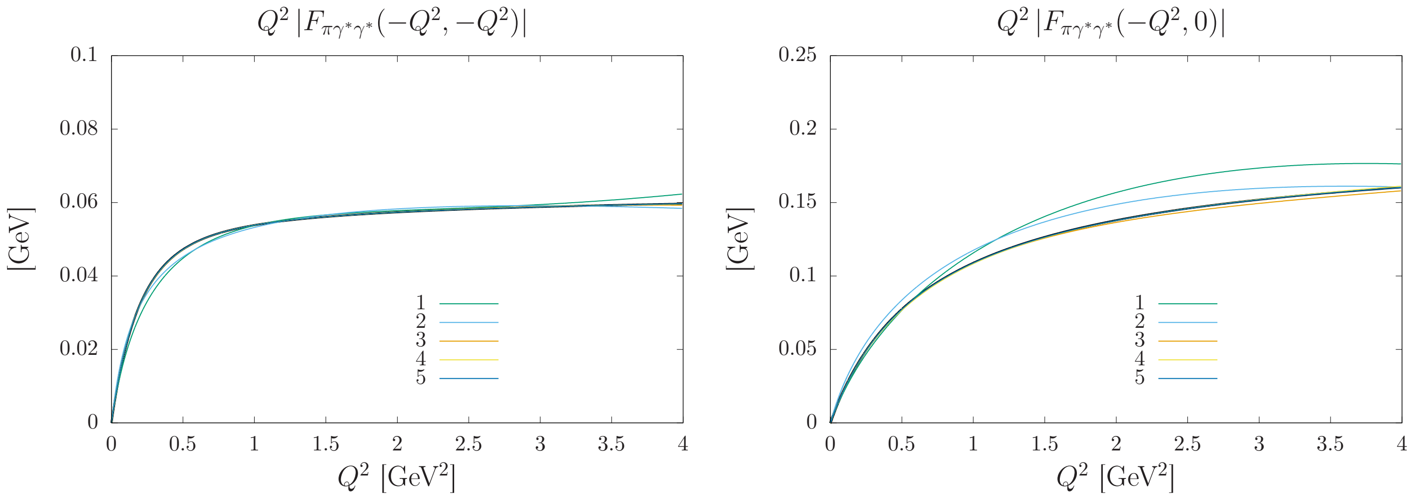
$\hookrightarrow$ Needs N relatively large to ensure good convergence

z-expansion inspired fits

- Fit the LMD or LMD+V model using a double z-expansion

  ↪ Estimate the systematic from a finite N

- Results for the LMD+V model with $Q^2_{\rm max} = 4~{\rm GeV}^2$:



↪ $N = 3$ and $Q^2_{\rm max} = 4~{\rm GeV}^2$ is already sufficient to get a precision bellow 1 % for the TFF

↪ The anomaly is recovered with a precision of 2 %

- We are currently testing this method with lattice QCD data
- For the pion-pole contribution HLbL ($a_\mu^{{\rm HLbL};\pi^0}$) : try to integrate data directly

Conclusion

- We have published a lattice calculation of the pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ with two dynamical quarks and in the momentum region relevant for the $(g-2)_\mu$.

- The LMD and LMD+V models describe our data successfully.

  → In particular we recover the anomaly prediction ($\alpha^{\rm th} = 0.274 \ \mathrm{GeV}^{-1}$) in the continuum and chiral limit

$$\alpha^{\rm LMD} = 0.275(18)(3) \ \mathrm{GeV}^{-1} \quad , \quad \alpha^{\rm LMD+V} = 0.273(24)(7) \ \mathrm{GeV}^{-1}$$

  → $7-9\%$ accuracy

- Disconnected contributions have been computed on one lattice ensemble.

- Provides a first lattice estimate of the pion-pole contribution $a_{\mu;\rm LMD+V}^{\rm HLbL;\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$

- We are now using the new CLS ensembles with $2+1$ dynamical quarks

  → Full $\mathcal{O}(a)$-improvement to reduce discretisation effects, dedicated study of finite-size effects

  → New kinematics included, higher statistics

  → Model independant extraction of $a_\mu^{\rm HLbL;\pi^0}$

- Use the results to study finite-size effect correction for the full HLbL lattice calculation on the lattice (see previous talks by Andreas Nyffeler and Nils Asmussen)