Neutral pion decay and the chiral anomaly on the lattice

Shoji Hashimoto (KEK)

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計算基礎科学連携拠点
Joint Institute for
JICFuS Computational Fundamental Science

国立大学法人
総合研究大学院大学
The Graduate University for Advanced Studies [SOKENDAI]

The $\pi^0 \rightarrow \gamma\gamma$

- Brief history…
  - Too small (Sutherland-Veltman (1967)).
    - Soft-pion relation, assuming PCAC
  - Triangle anomaly (Adler (1969), Bell-Jackiw (1969))
    - Anomalous term in Ward-Takahashi identity
  - One-loop is exact (Adler-Bardeen (1969)).

- Renewed interest
  - Form factor away from the soft-pion limit → HLbL

Lattice calculation?

Non-trivial, because…

1. Chiral symmetry is the key for axial anomaly.
   - Chiral symmetry is a central issue for lattice fermions.
   - Want to maintain, but have to reproduce anomaly.

2. Final state is not a QCD eigenstate.
   - If we do naively, we get $\pi^0 \rightarrow \rho\rho$.
   - Need to fix four-momentum.

Plan

1. Anomaly and lattice
   - $\pi^0 \rightarrow \gamma\gamma$ and axial anomaly
   - Lattice fermions

2. Method for photon external states
   - How to treat non-QCD final states

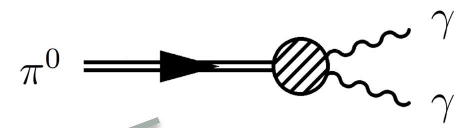
3. An old calculation
   - With the overlap fermion

4. Discussions

1. Anomaly and lattice

- $\pi^0 \rightarrow \gamma\gamma$ and axial anomaly
- Lattice fermions

Standard argument



- $\pi^0 \to \gamma\gamma$ comes from axial anomaly.

$$\Gamma_{\mu\nu}(k_1,k_2,q) = e^2 \int d^4z d^4y\, e^{ik_1 z + ik_2 y} \left\langle 0 \left| T\left(J_\mu^{em}(z) J_\nu^{em}(y)\right) \right| \pi^0(q) \right\rangle$$

$$\Gamma_{\mu\nu\lambda}(k_1,k_2,q) = e^2 \int d^4x d^4y\, e^{ik_2 y - iqx} \left\langle 0 \left| T\left(J_\mu^{em}(0) J_\nu^{em}(y) A_\lambda^3(x)\right) \right| 0 \right\rangle$$

$$q^\lambda \Gamma_{\mu\nu\lambda}(k_1,k_2,q) = -ie^2 \int d^4x d^4y\, e^{ik_2 y - iqx} \left\langle 0 \left| T\left(J_\mu^{em}(0) J_\nu^{em}(y) \partial^\lambda A_\lambda^3(x)\right) \right| 0 \right\rangle$$

vanish in the q→0 limit.

$= f_\pi m_\pi^2 \phi^3(x)$

$$= \frac{f_\pi m_\pi^2}{m_\pi^2 - q^2} \Gamma_{\mu\nu}(k_1,k_2,q) \quad - \frac{ie^2}{12\pi^2} \varepsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma$$

- Sutherland-Veltman (1967): vanish in the soft pion limit.
- Not really true, because

$$\partial^\lambda A_\lambda(x) = 2mP(x) - \frac{e^2}{2\pi^2} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)$$

Adler-Bell-Jackiw anomaly

Lattice gauge theory

- The way that axial anomaly realizes depends on the regularization:
  - Wilson fermion:
    - Chiral symmetry is explicitly violated.
    - No conserved current. $\partial_\mu A_\mu - 2mP \neq 0$ coincides with the chiral anomaly in the continuum limit (Karsten-Smit (1981)).
  - Ginsparg-Wilson fermions: $$\delta\bar{\psi} = i\alpha\bar{\psi}\left(1-\frac{a}{2\rho}D\right)\gamma_5,\ \delta\psi = i\alpha\gamma_5\left(1-\frac{a}{2\rho}D\right)\psi$$
    - Modified chiral symmetry:
    - Axial-current conserves, classically. In the quantum theory, the fermion measure produces Jacobian, which leads to the correct anomaly (as in Fujikawa's analysis).
    - Shown in perturbation theory (Kikukawa-Yamada (1998)).

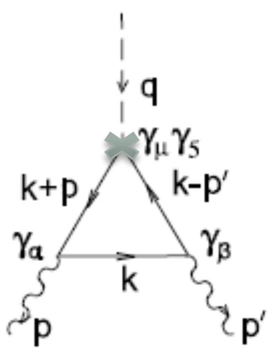
$\pi^0 \rightarrow \gamma\gamma$ with Wilson fermion

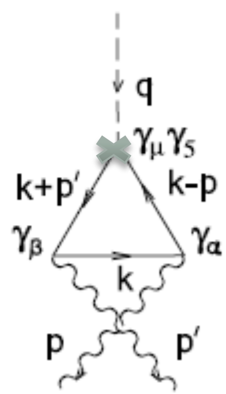
- Axial Ward identity

$$\Delta_\mu^L J_\mu^{(5)}(x) = 2m\overline{\psi}(x)\gamma_5\psi(x) + \underline{X(x)}$$

term to describe the violation of chiral symmetry ← from the Wilson term

- Triangle diagram





Continuum:
- Linear divergence. Arbitrary shift of momentum not allowed.
- Anomaly is from UV…?

Wilson fermion:
- Only the term of X(x) survives.

$\pi^0 \to \gamma\gamma$ with Wilson fermion

- After some algebra… Karsten-Smit (1983), Rothe-Sadooghi (1998)

$$\sim \varepsilon_{\mu\nu\rho\lambda} p_\rho p'_\lambda \int_{-\pi}^{\pi} \frac{d^4 q}{(2\pi)^4} \frac{\cos q_\mu \cos q_\nu \cos q_\rho (\mathcal{M}^2(q) \cos q_\lambda - 4r\mathcal{M}(q) \sin^2 q_\lambda)}{[\mathcal{M}^2(q) + \sum_\sigma \sin^2 q_\sigma]^3}$$

$$\mathcal{M}(q) = m + r \sum_\sigma [1 - \cos q_\sigma]$$

$$\to \varepsilon_{\mu\nu\rho\lambda} p_\rho p'_\lambda \int_0^\infty q^2 dq^2 \frac{(m + \cancel{rq^2/2})^2 - 4r(m + \cancel{rq^2/2})q^2}{[(m + \cancel{rq^2/2})^2 + q^2]^3}$$

integral strongly peaked at $q^2 \sim m^2$

- ABJ anomaly actually comes from IR.
- Only when m~0 in the massless limit, which is non-trivial.

$\pi^0 \rightarrow \gamma\gamma$ with overlap fermion

- Exact chiral symmetry on the lattice

$$\delta\bar{\psi} = i\alpha\bar{\psi}\left(1 - \frac{a}{2\rho}D\right)\gamma_5,\ \delta\psi = i\alpha\gamma_5\left(1 - \frac{a}{2\rho}D\right)\psi$$

  - ABJ anomaly arises from the Jacobian of the path integral measure (like in Fujikawa's method)

$$\frac{1}{2}\mathrm{Tr}[\gamma_5 D] = \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

- Then, $\pi^0 \rightarrow \gamma\gamma$ arises from $2mP = \cancel{\partial_\mu A_\mu} - FF\sim$

2. Method for photon external states

- How to treat non-QCD final states

Non-QCD final state

- First guess: $\pi^0 \to \gamma\gamma$ is extracted from a PVV 3-point func.
  - Correct. But, doesn't work if too naïve.
  - At large (Euclidean) time separations, PVV corresponds to $\pi^0 \to \rho\rho$, though unphysical.
- More careful derivation:

  Ji, Jung, PRL 86 (2001) 208; Dudek, Edwards, PRL 97 (2006) 172001.

  - LSZ reduction formula

$$
\langle \gamma(p_1,\lambda_1)\gamma(p_2,\lambda_2) | \pi(q) \rangle = - \lim_{\substack{p_1' \to p_1 \\ p_2' \to p_2}} \varepsilon_\mu^*(p_1,\lambda_1)\varepsilon_\nu^*(p_2,\lambda_2) p_1'^2 p_2'^2
$$

$$
\times \int d^4x\, d^4y\, e^{ip_1'y + ip_2'x} \langle 0 | T\left\{ A^\mu(y) A^\nu(x) \right\} | \pi(q) \rangle
$$

photon fields

Non-QCD final state

- Perturbative expansion for QED

$$\langle \gamma(p_1,\lambda_1)\gamma(p_2,\lambda_2)|\pi(q)\rangle = -e^2 \lim_{\substack{p_1'\to p_1 \\ p_2'\to p_2}} \varepsilon_\mu^*(p_1,\lambda_1)\varepsilon_\nu^*(p_2,\lambda_2) p_1'^2 p_2'^2$$

$$\times \int d^4x\, d^4y\, d^4w\, d^4z\, e^{ip_1'y+ip_2'x} D^{\mu\rho}(y,z) D^{\nu\sigma}(x,w) \langle 0|T\{j_\rho(z) j_\sigma(w)\}|\pi(q)\rangle$$

photon propagator

$$D^{\mu\nu}(0,z) = -ig^{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ikz}}{k^2+i\varepsilon}$$

vector current $j_\mu(x) = \overline{q}(x)\gamma_\mu q(x)$

canceling $p_1'^2 p_2'^2$

  - It reduces to

$$\int d^4x\, e^{ip_1x} \langle 0|T\{j^\mu(x) j^\nu(0)\}|\pi(q)\rangle$$

Non-QCD final state

- Analytic continuation

$$\int d^4x\, e^{ip_1 x} \langle 0 | T\left\{ j^\mu(x) j^\nu(0) \right\} | \pi(q) \rangle$$

$$\Rightarrow \int dt\, e^{\omega t} \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \langle 0 | T_E\left\{ j_\mu(x) j_\nu(0) \right\} | \pi(q) \rangle$$

  - Possible only when p_1^2 does not develop singularities (cut, pole) = photon does not mix with other QCD states.
  - Necessary to ensure that the photon field correlator is saturated by the photon state.
  - We are restricted in the region p_1^2, $p_2^2 < m_\rho^2$ (or $< 4E_\pi^2$).

Non-QCD final state

- On the lattice we calculate

$$M_{\mu\nu}(p_1,p_2) = \lim_{t_{1,2}-t_\pi\to\infty} \frac{1}{\frac{\phi_{\pi,\vec{q}}}{2E_{\pi,\vec{q}}} e^{-E_{\pi,\vec{q}}(t_2-t_\pi)}} \int dt_1\, e^{\omega(t_1-t_2)} C_{\mu\nu}(t_1,t_2,t_\pi),$$

$$C_{\mu\nu}(t_1,t_2,t_\pi) \equiv \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \int d^3\vec{z}\, e^{i\vec{q}\cdot\vec{z}} \left\langle 0 \middle| T_E\left\{ j_\mu(\vec{x},t_1) j_\nu(\vec{y},t_2) P(\vec{z},t_\pi)\right\} \middle| 0 \right\rangle$$

  - ω is an arbitrary parameter as far as $p_1 = (\omega, \vec{p}_1)$ satisfies $p_1^2 < m_\rho^2$.
  - p_2 is given by momentum conservation: $p_2 = (E_{\pi,\vec{q}} - \omega, \vec{q} - \vec{p}_1)$

  - The factor $e^{\omega(t_1-t_2)}$ seems divergent when $|t_1 - t_2|$ is large. Is it well-defined? Yes. The lowest-lying vector states survives and gives a stronger suppression factor $e^{-E_\rho(t_1-t_2)}$.

Form factor to be obtained

- Then, we obtain

$$M_{\mu\nu}(p_1,p_2) = i\int d^4x\, e^{ip_1x} \left\langle 0 \middle| T\left\{ j_\mu(x) j_\nu(0) \right\} \middle| \pi^0(q) \right\rangle$$
$$= \varepsilon_{\mu\nu\alpha\beta} p_1^\alpha p_2^\beta F_{\pi^0\gamma\gamma}(m_\pi^2; p_1^2, p_2^2)$$

- $F_{\pi\gamma\gamma}$ is the form factor of interest.
  - In the soft pion limit, we should reproduce the ABJ anomaly.

$$F_{\pi^0\gamma\gamma}(0;0,0) = \frac{1}{4\pi^2 F_0}$$

  - Away from the chiral limit, it gives the correction due to finite quark mass.
  - For off-shell photons $p_{1,2}{}^2 \neq 0$, it gives a form factor relevant for other process like $\gamma^*\pi^0 \rightarrow \gamma$.

3. An old calculation

- Dynamical overlap fermion (not in detail)
- All-to-all propagator

First look at the problem

Xu Feng et al. [JLQCD collaboration], PRL109, 182001 (2012); arXiv:1206.1375.

1. Test if the overlap fermion formulation really produces the (theoretically required) chiral anomaly in the physical process.

2. Calculate the $\pi^0 \rightarrow \gamma\gamma$ amplitude away from the limit of massless pion and soft photons. Only some phenomenological analyses (χPT, etc) are available so far.

Dynamical overlap fermion

- Ensemble generation (JLQCD collaboration, 2006~)
  - 2+1 flavors represented by overlap fermions
  - Lattice spacing a = 0.11 fm (Iwasaki gauge action).
  - Lattice volume: 16^3x48, 24^3x48.
  - Quark mass: 4 points covering m_π = 290 – 540 MeV.

  - Gauge field topology fixed. (Necessary to run dynamical overlap reasonably fast.) A source of finite volume effect of O(1/V).

  - Was successful to calculate the Dirac spectrum and to extract chiral condensate Σ. Several other applications.

All-to-all propagator

- Quark propagator from any y to any x.

$$D^{-1}(x,y) = \sum_{k=1}^{N_{ev}} \frac{1}{\lambda^{(k)}} u^{(k)}(x) u^{(k)\dagger}(y) + \sum_{d=1}^{N_d} \left[D_{high}^{-1} \eta^{(d)} \right](x) \eta^{(d)}(y)$$

Low mode contribution

$$Du^{(k)}(x) = \lambda^{(k)} u^{(k)}(x)$$

High mode contribution
From a random noise

Random noise

  - Low-lying eigenvalues/eigenvectors are calculated and stored.
  - Also used to accelerate the fermion inversion (exact deflation).
  - High-mode contribution is calculated and stored. N_d times = each color/spinor, (every other) time-slice.

Foley et al., Comp. Phys. Comm. 172 (2005) 145.
and references therein.

All-to-all propagator

- To calculate

$$M_{\mu\nu}(p_1,p_2) = \lim_{t_{1,2}-t_\pi\to\infty} \frac{1}{\frac{\phi_{\pi,\vec{q}}}{2E_{\pi,\vec{q}}} e^{-E_{\pi,\vec{q}}(t_2-t_\pi)}} \int dt_1\, e^{\omega(t_1-t_2)} C_{\mu\nu}(t_1,t_2,t_\pi),$$

$$C_{\mu\nu}(t_1,t_2,t_\pi) \equiv \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \int d^3\vec{z}\, e^{i\vec{q}\cdot\vec{z}} \left\langle 0 \middle| T_E\left\{ j_\mu(\vec{x},t_1) j_\nu(\vec{y},t_2) P(\vec{z},t_\pi)\right\} \middle| 0 \right\rangle$$

  - 3pt functions with all x, y, z points have to be summed. Different time-slices with different weight depending on ω.

- Momentum configuration taken:
  - #1: p_1 = (0,0,0), q = (0,0,1)
  - #2: p_1 = (0,0,1), q = (0,0,0)

in unit of 2π/L

Analysis steps

- A bit different from conventional lattice analysis:

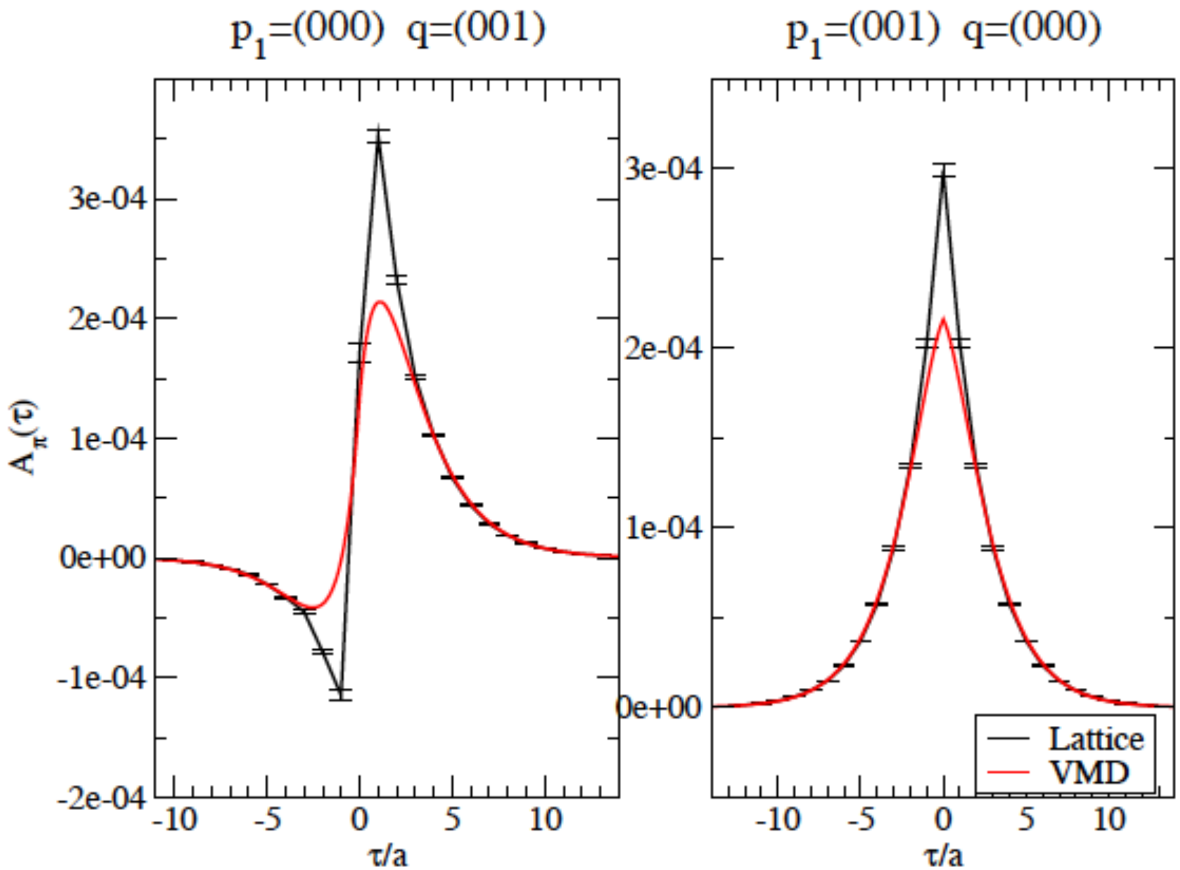
$$M_{\mu\nu}(p_1,p_2) = \lim_{t_{1,2}-t_\pi\to\infty} \frac{1}{\dfrac{\phi_{\pi,\vec{q}}}{2E_{\pi,\vec{q}}} e^{-E_{\pi,\vec{q}}(t_2-t_\pi)}} \int dt_1\, e^{\omega(t_1-t_2)} C_{\mu\nu}(t_1,t_2,t_\pi),$$

$$C_{\mu\nu}(t_1,t_2,t_\pi) \equiv \int d^3\vec{x}\, e^{-i\vec{p}_1\cdot\vec{x}} \int d^3\vec{z}\, e^{i\vec{q}\cdot\vec{z}} \left\langle 0 \middle| T_E\left\{ j_\mu(\vec{x},t_1) j_\nu(\vec{y},t_2) P(\vec{z},t_\pi) \right\} \middle| 0 \right\rangle$$

  - Need an integral over t_1.
  - First, we want to see how $C_{\mu\nu}(t_1,t_2,t_\pi)$ looks like.
  - Define

$$A_\pi(\tau) \equiv \lim_{t-t_\pi\to\infty} \frac{C_{\mu\nu}(t_1,t_2,t_\pi)}{e^{-E_{\pi,\vec{q}}(t-t_\pi)}}, \quad \tau = t_1 - t_2,\ t = \min\{t_1,t_2\}$$

which is the integrand.

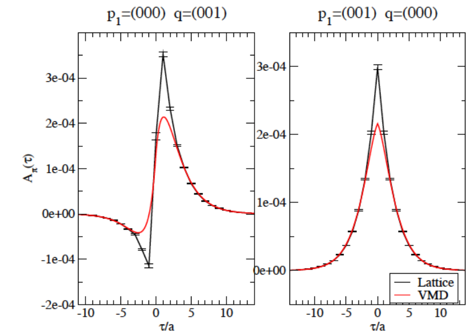


- As a cross check, a VMD curve is also shown.

$$F^{VMD}_{\pi^0\gamma\gamma}(m_\pi^2; p_1^2, p_2^2) = c_V G_V(p_1^2) G_V(p_2^2), \quad G_V(p^2) = \frac{m_V^2}{m_V^2 - p^2}$$

- When $|t_1 - t_2|$ is large, the lowest vector state saturates.

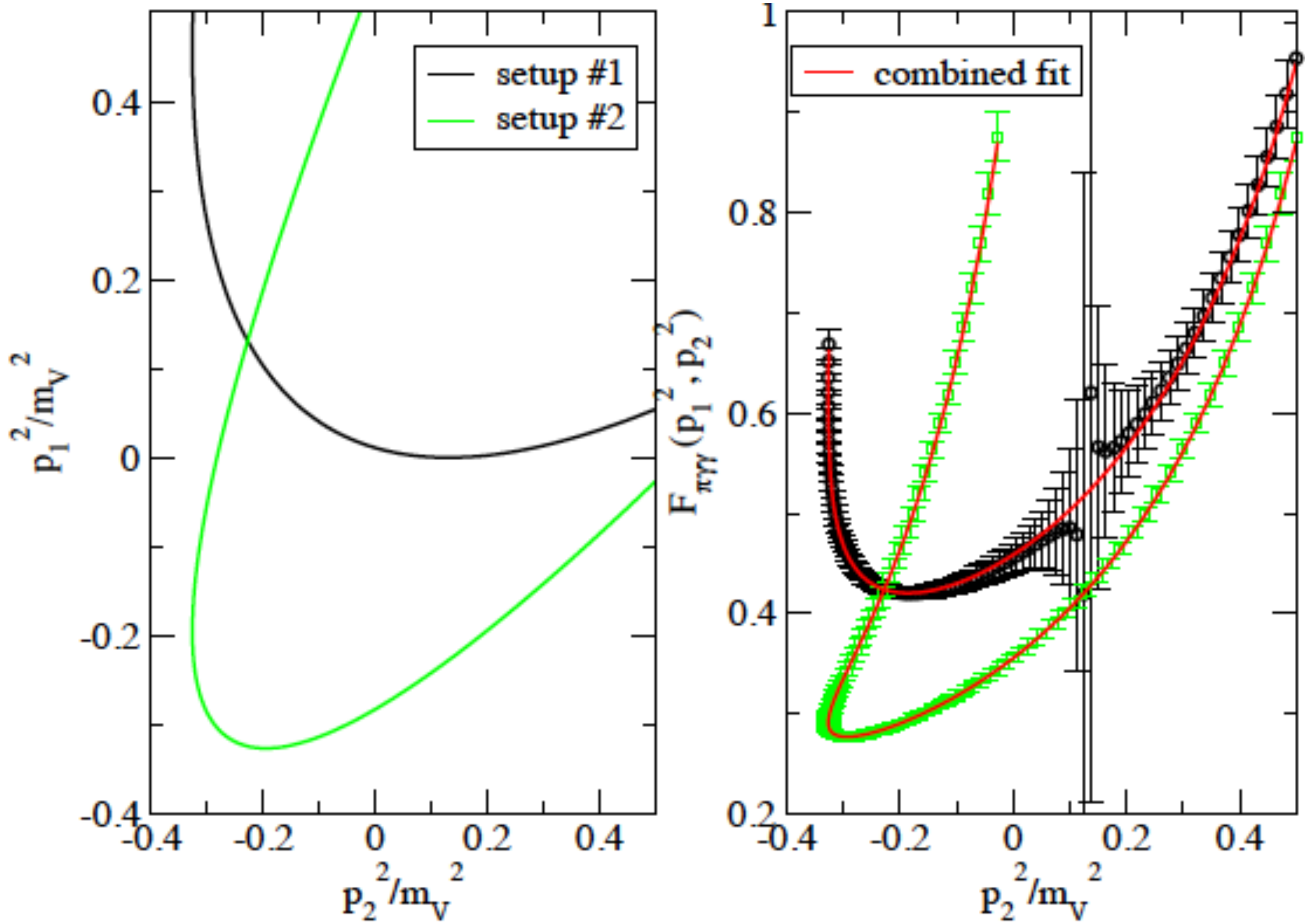
Integral



- When integrate over t_1,
  - We sum over the lattice data at short distances.
  - Long distances are replaced by $\exp(-M_V|\tau|)$ and summed.

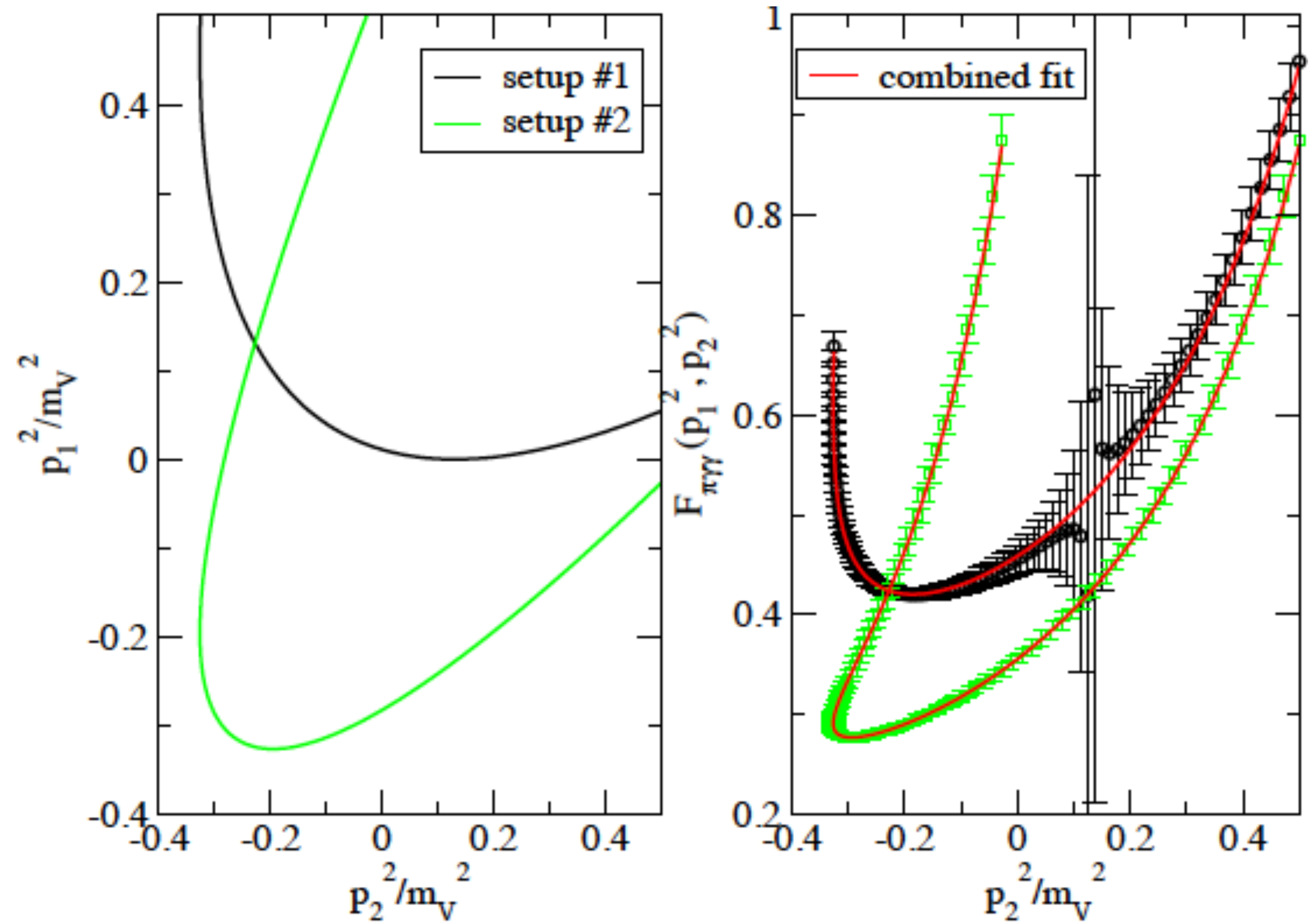
  - Worry about discretization effects? Checked with the VMD curve: the difference between sum and integral is only 0.05%. Expect larger for the real thing, but would be ~0.1%. Tiny.

- Obtain $M_{\mu\nu}(p_1^2,p_2^2)$ by multiplying $e^{\omega\tau}$ and integrate.
  - Results on a continuous curve on the (p_1^2,p_2^2) plane.



- Not just at a point, but on a curve satisfying

$$(p_1^2, p_2^2) = (\omega^2 - \vec{p}_1^{\,2}, (E_{\pi,\vec{q}} - \omega)^2 - (\vec{q} - \vec{p}_1)^2)$$



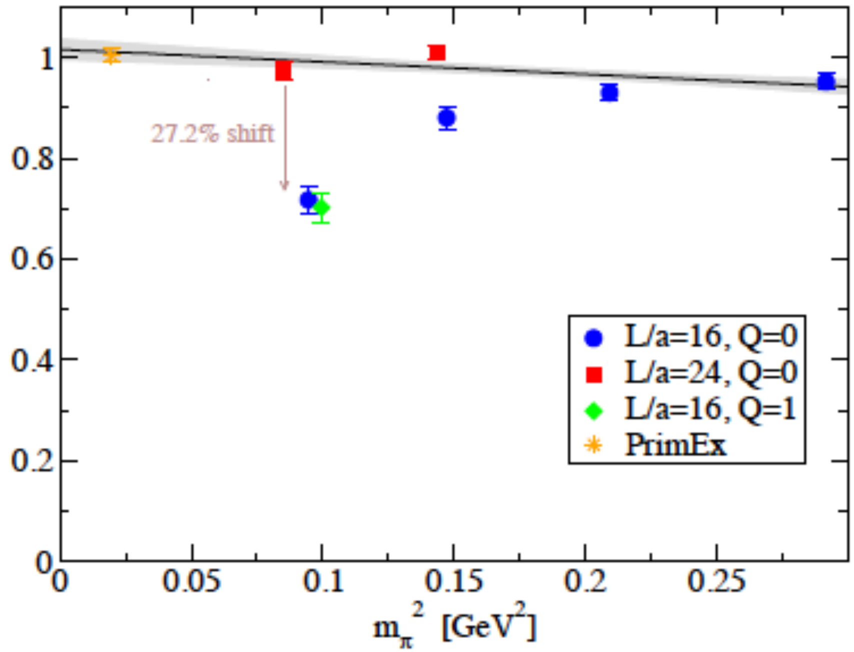
- Fit curve represents a function of the form

$$F_{\pi^0\gamma\gamma}(m_\pi^2; p_1^2, p_2^2) = c_V G_V(p_1^2) G_V(p_2^2) + \sum_m c_m \left((p_2^2)^m G_V(p_1^2) + (p_1^2)^m G_V(p_2^2) \right) + \sum_{m,n} c_{m,n} (p_1^2)^m (p_2^2)^n$$

but only c_V, c_0, $c_{0,0}$ and $c_{0,1}$

Test against ABJ

$$\frac{F_{\pi^0\gamma\gamma}(m_\pi^2;0,0)}{F^{ABJ}_{\pi^0\gamma\gamma}}$$



- Data extrapolated to the on-shell photon limit $p_1{}^2=p_2{}^2=0$.
- Data with $m_\pi L > 4$ are consistent with ABJ and exp.

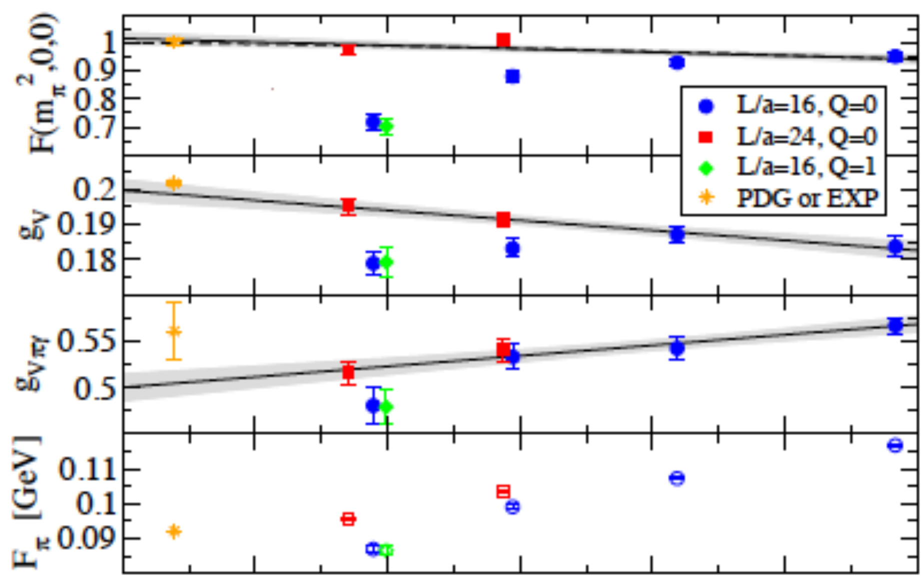
Large finite volume effect?

Finite volume effect?

- JLQCD lattices were too small…
  - FVE was seen in other related modes.
  - Try to see the effect in individual component.
  - Reasonable to add up to the big effect.
  - Correction is attempted based on the data for each component.

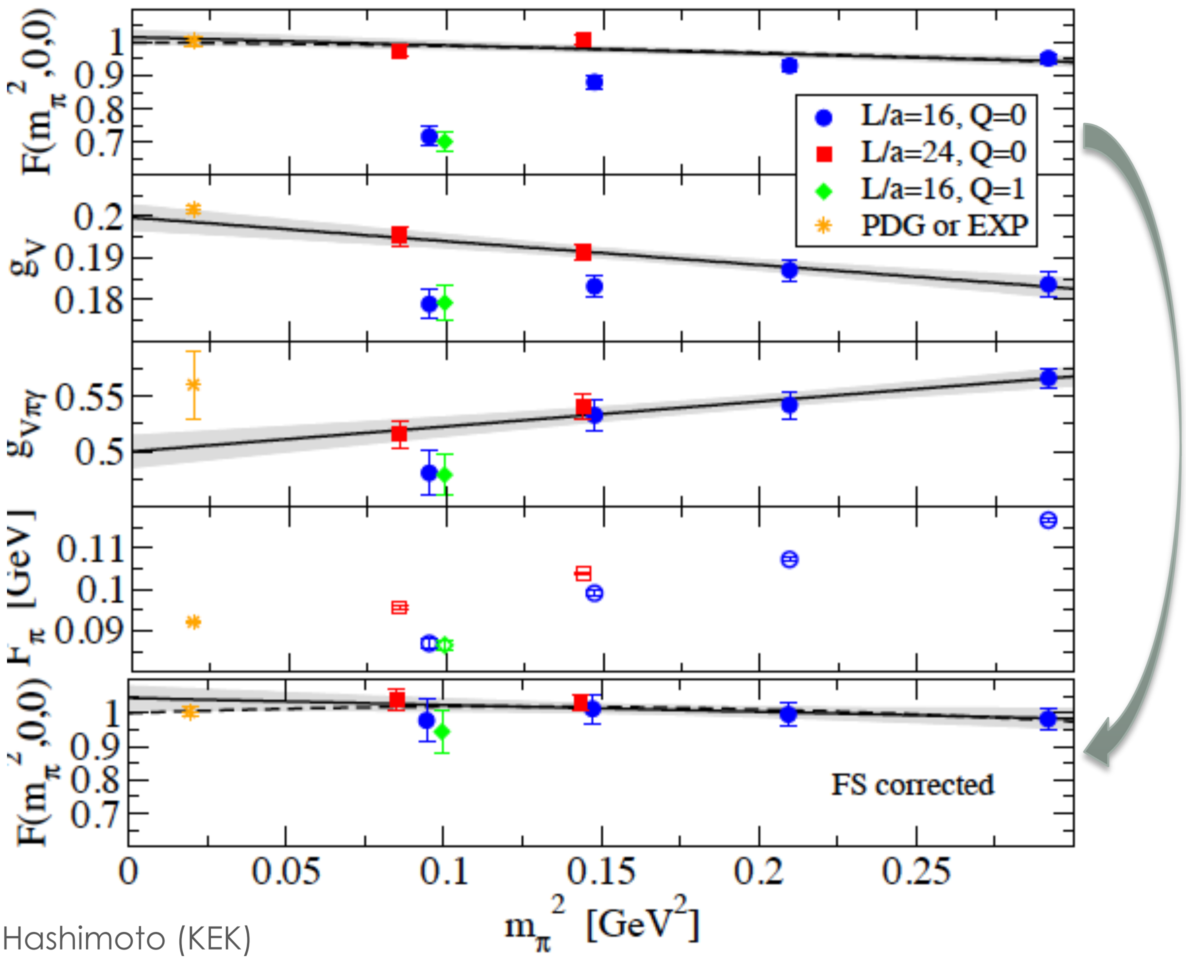
$$\langle j_\mu j_\nu P\rangle \sim \langle 0|j_\mu|V\rangle\langle V|j_\nu|\pi\rangle\langle\pi|P|0\rangle$$

$$\sim g_V \times g_{V\pi\gamma} \times F_\pi$$



Finite volume effect?

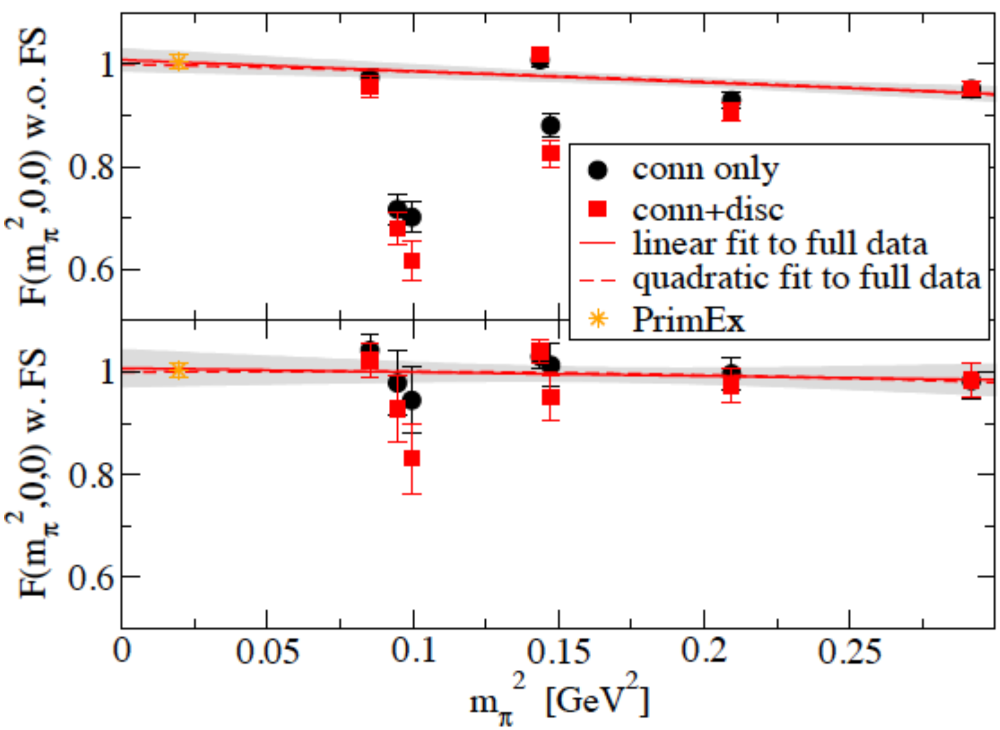
- Assuming that FVE has the form $\exp(-m_\pi L)$.

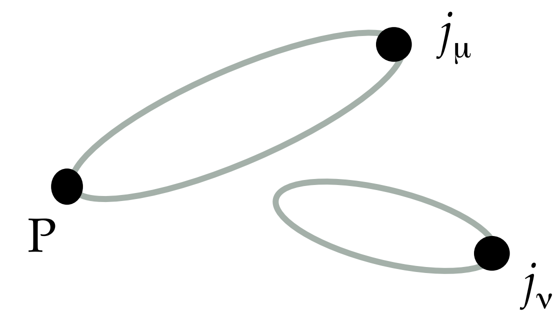


seems to be consistent.

Disconnected contribution?

- Often ignored, but with all-to-all, straightforward to include the disconnected diagrams.





- Vanishes in the SU(3) limit (2/3-1/3-1/3).
- Small as expected. But, will become important at the per cent level.

Numerical results

- After examining the systematic effects, we obtain

$$
\begin{aligned}
F(0,0,0) &= 1.009(22)(29)\ , \\
F(m_{\pi,\mathrm{phy}}^2,0,0) &= 1.005(20)(30)\ , \\
\Gamma_{\pi^0\gamma\gamma} &= 7.83(31)(49)\ \mathrm{eV}
\end{aligned}
$$

form factor normalized by ABJ.

  - Consistent with the ABJ anomaly.
  - Quark mass dependence insignificant.

  - Also consistent with PrimEX: 7.82(22) eV.

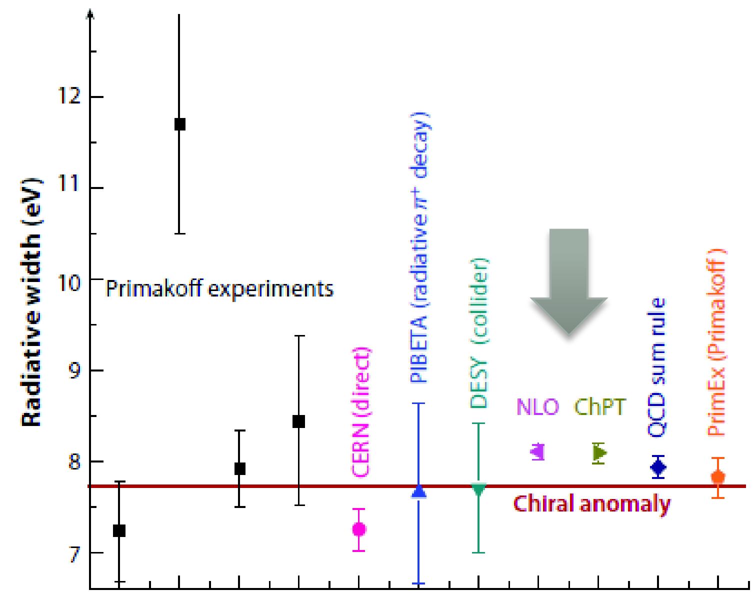
4. Discussions

Consistent with pheno?

- Several works predicted about +(4—5)% effects. Consistent?

  Review article on ph and ex:
  Bernstein, Holstein, arXiv:1112.4890.

  - Sum rule, χPT, …
  - Mainly due to a mixing of π^0 with η and η'.



More realistic calculation must take isospin breaking into account.
Overall understanding including η and η'.

Other applications?

- Lattice calculation of $\pi^0 \rightarrow \gamma\gamma$ is possible. Anomaly is reproduced. Non-QCD final state can be treated. Other interesting application?
  - $\pi^0 \rightarrow e^+e^-$: through $\pi^0 \rightarrow \gamma^*\gamma^*$. Requires info on the off-shell form factor.
  - Muon g-2: Light-by-light would be dominated by $\gamma\gamma^* \rightarrow \pi^0 \rightarrow \gamma^*\gamma^*$.
  - $\gamma^*\pi^0 \rightarrow \gamma$: Going to higher momentum transfer ($\rightarrow$ Mainz group). Experimental data available. Eventually connects to the perturbative regime of Brodsky-Lepage.

Conclusions

- Chiral lattice fermion works just as expected.
  - Well, more expensive. Yet, theoretically clean formulation is helpful when new applications are considered.
  - Wilson fermion seems to work well, too. See Andreas' talk.
- Non-QCD initial/final state can be treated.
  - May allow more applications.