π^0, η, and η' transition form factors

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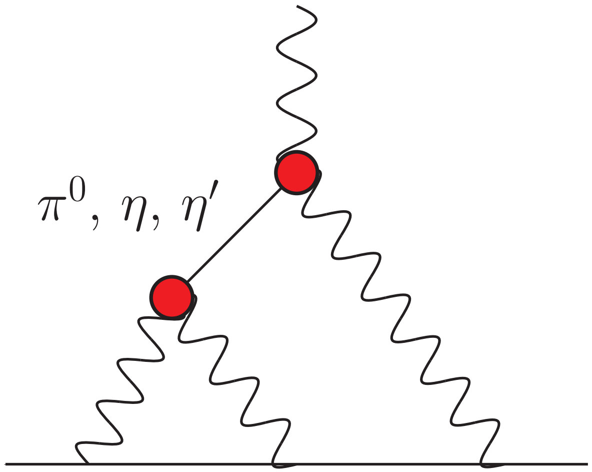
UConn, 13/3/2018

Dispersion relations for meson transition form factors

Ingredients for a data-driven analysis of $\pi^0, \eta, \eta' \to \gamma^*\gamma^{(*)}$

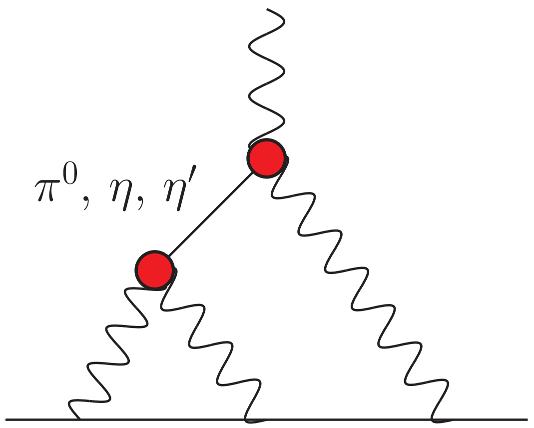
- summary: from $\gamma\pi \to \pi\pi$ via $e^+e^- \to 3\pi$ to $\pi^0 \to \gamma^*\gamma$
  see M. Hoferichter 2017
- new: implementation of high-energy asymptotics
  first preliminary results: doubly-virtual, $(g-2)_\mu$
- summary: from radiative decays $\eta^{(\prime)} \to \pi^+\pi^-\gamma$
  to (singly-virtual) transition form factors $\eta^{(\prime)} \to \gamma^*\gamma$ see BK 2017
- factorization breaking for
  doubly-virtual form factor:
  $e^+e^- \to \eta\pi^+\pi^-$, $\eta' \to \pi^+\pi^-\pi^+\pi^-$



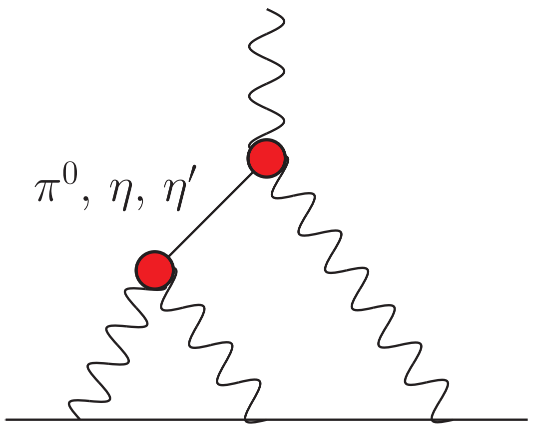
Summary / Outlook

Pseudoscalar transition form factors and $(g-2)_\mu$

- largest individual HLbL contribution:
  pseudoscalar pole terms
  singly / doubly virtual form factors
  $F_{P\gamma\gamma^*}(q^2, 0)$ and $F_{P\gamma^*\gamma^*}(q_1^2, q_2^2)$



Pseudoscalar transition form factors and $(g-2)_\mu$



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- normalisation fixed by Wess–Zumino–Witten anomaly, e.g.:

$$F_{\pi^0\gamma\gamma}(0,0) = \frac{e^2}{4\pi^2 F_\pi}$$

F_π: pion decay constant $\longrightarrow$ measured at 1.4% level PrimEx 2011
planned at 0.7% level (preliminary: 0.85%) PrimEx 2015, in progress

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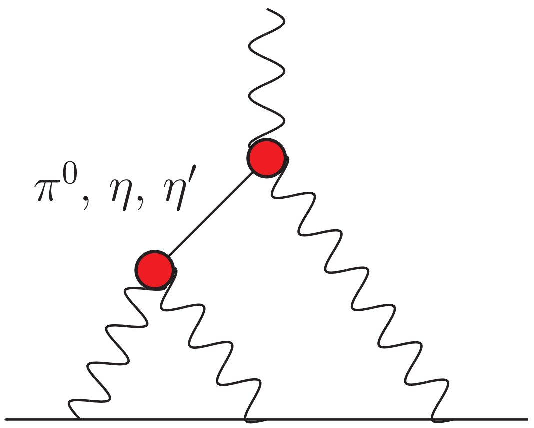

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- q_i^2-dependence: often modelled by vector-meson dominance
  $\longrightarrow$ what can we learn from analyticity and unitarity constraints?
  $\longrightarrow$ what experimental input sharpens these constraints?

Dispersive analysis of $\pi^0/\eta \to \gamma^*\gamma^*$

- isospin decomposition:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{vs}(q_1^2,q_2^2) + F_{vs}(q_2^2,q_1^2)$$
$$F_{\eta\gamma^*\gamma^*}(q_1^2,q_2^2) = F_{vv}(q_1^2,q_2^2) + F_{ss}(q_1^2,q_2^2)$$

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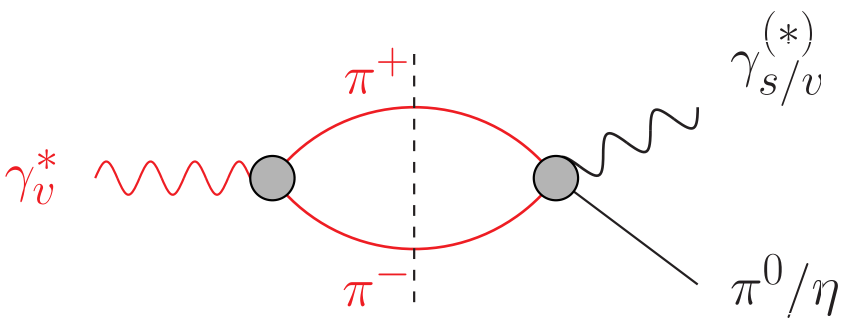
- analyse the leading hadronic intermediate states:

Hanhart et al. 2013, Hoferichter et al. 2014



▷ isovector photon: 2 pions

$\propto$ pion vector form factor $\times$ $\gamma\pi \to \pi\pi$ / $\eta \to \pi\pi\gamma$

all determined in terms of pion–pion P-wave phase shift

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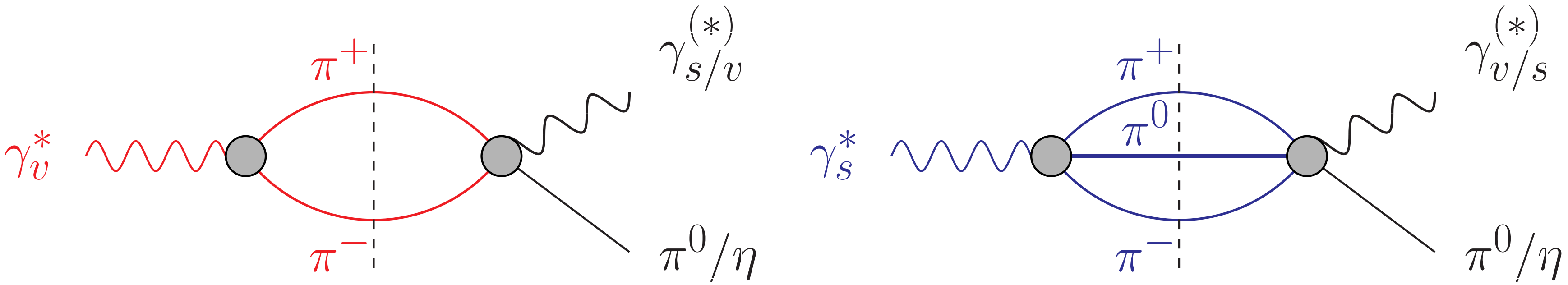

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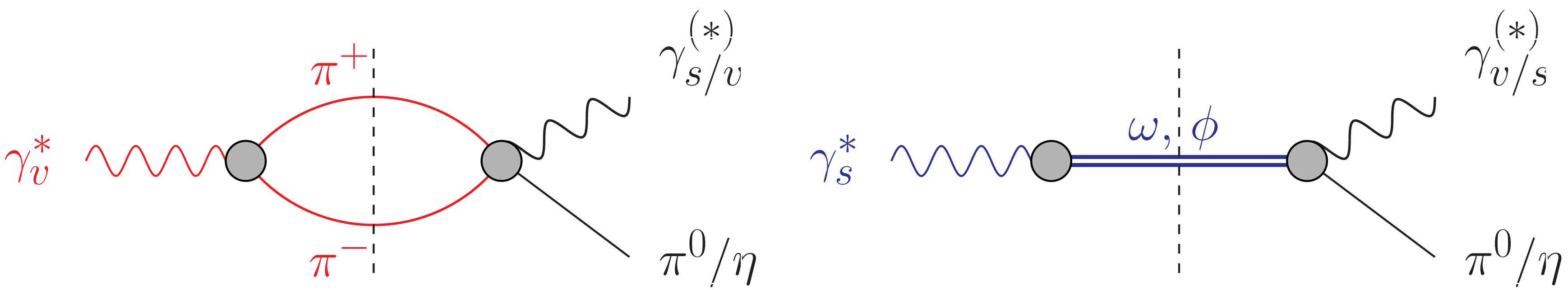

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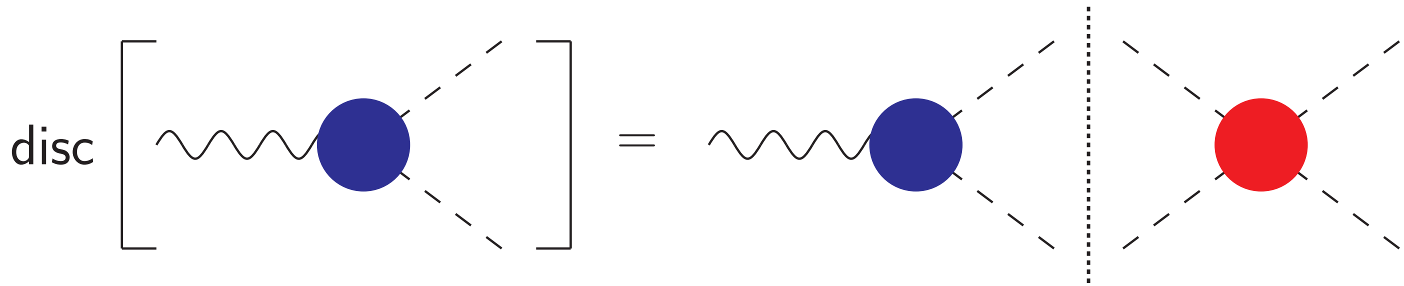
$\propto$ pion vector form factor $\times$ $\gamma\pi \to \pi\pi$ / $\eta \to \pi\pi\gamma$

all determined in terms of pion–pion P-wave phase shift

▷ isoscalar photon: 3 pions $\longrightarrow$ dominated by narrow $\omega,\ \phi$

$\leftrightarrow \omega/\phi$ transition form factors; very small for the η

Charged pion form factor: Omnès representation



$$\frac{1}{2i}\,\text{disc}\,F_\pi^V(s) = \text{Im}\,F_\pi^V(s) \;=\; F_\pi^V(s)\times\theta(s-4\,M_\pi^2)\times\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

$\longrightarrow$ final-state theorem: phase of $F_\pi^V(s)$ is just $\delta_1^1(s)$ Watson 1954

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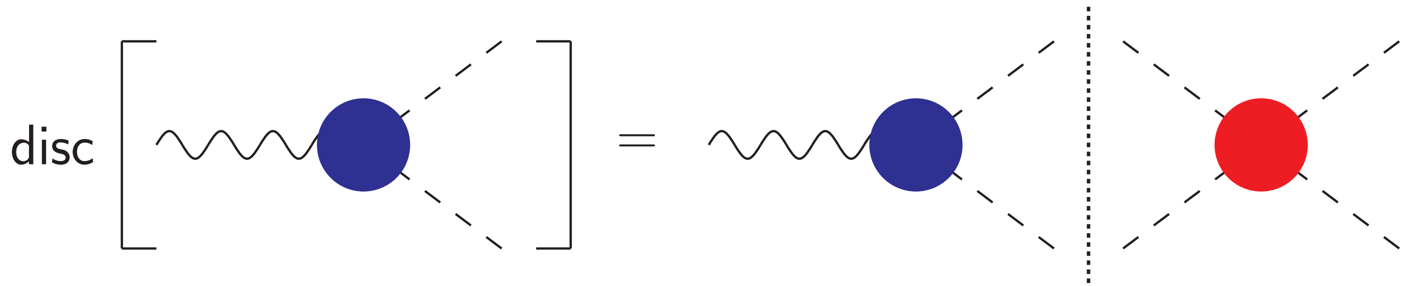

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- solution:

$$F_\pi^V(s) = P(s)\Omega(s) \; , \quad \Omega(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

$P(s)$ polynomial, $\Omega(s)$ Omnès function Omnès 1958

▷ $\pi\pi$ phase shifts from Roy equations

Ananthanarayan et al. 2001, García-Martín et al. 2011

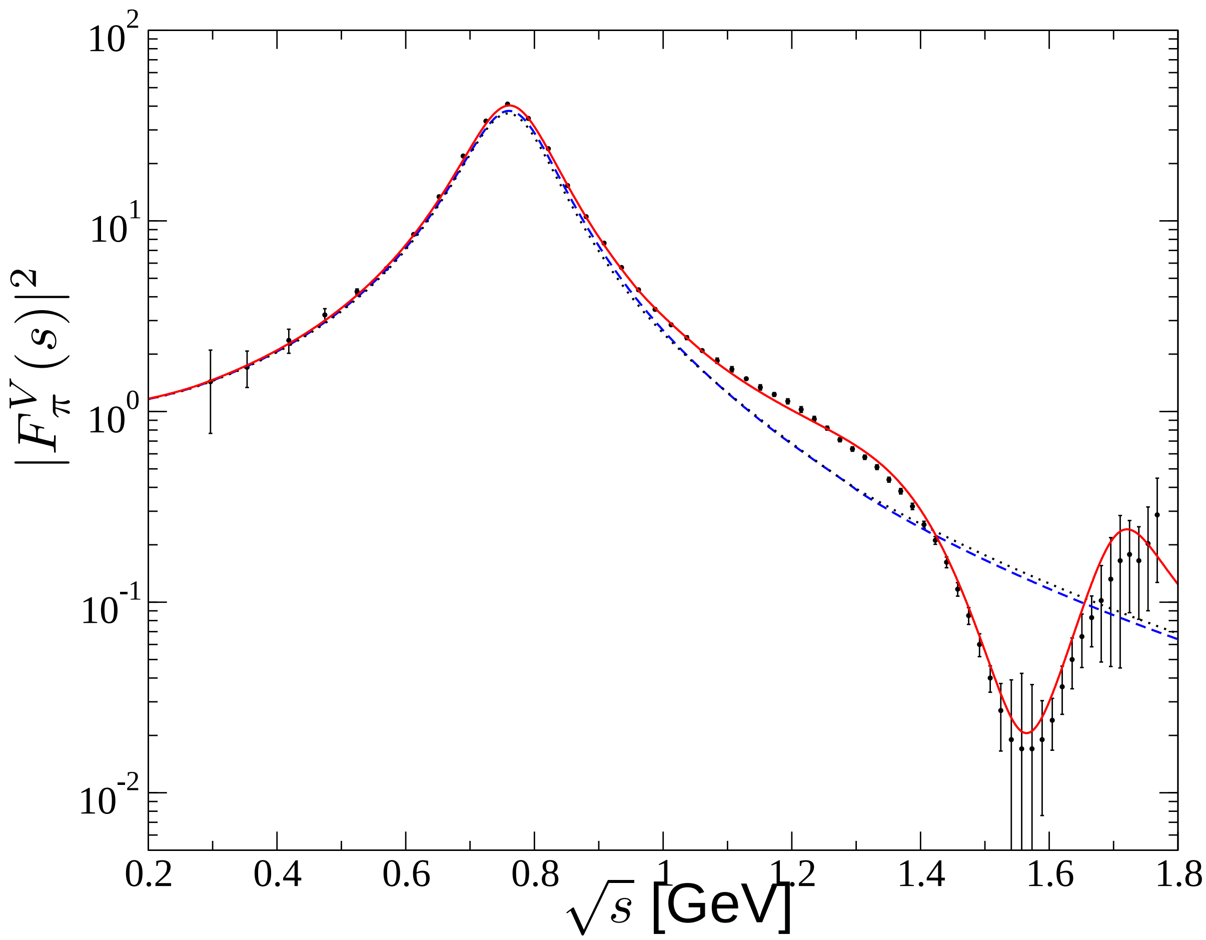
▷ $P(0) = 1$ from symmetries (gauge invariance)

- below 1 GeV: $F_\pi^V(s) \approx (1 + 0.1\,\text{GeV}^{-2} s)\Omega(s)$

slope due to inelastic resonances ρ', ρ''... Hanhart 2012

Pion vector form factor vs. Omnès representation

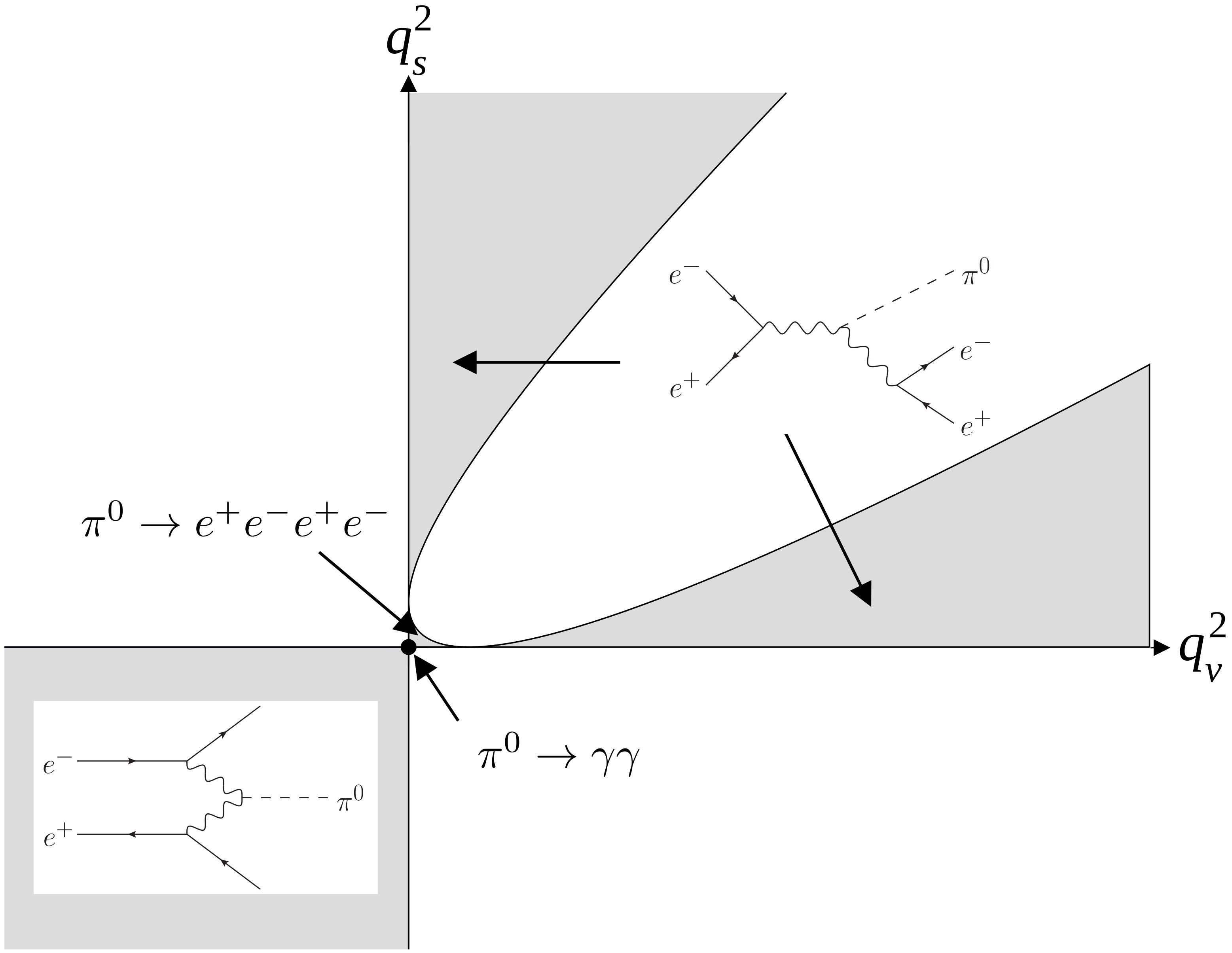
Data on pion form factor in $\tau^- \to \pi^- \pi^0 \nu_\tau$ Belle 2008



$\pi\pi$ P-wave phase shift / effective form factor phase incl. ρ', ρ''

Schneider et al. 2012

$\pi^0 \to \gamma^*(q_v^2)\gamma^*(q_s^2)$ transition form factor



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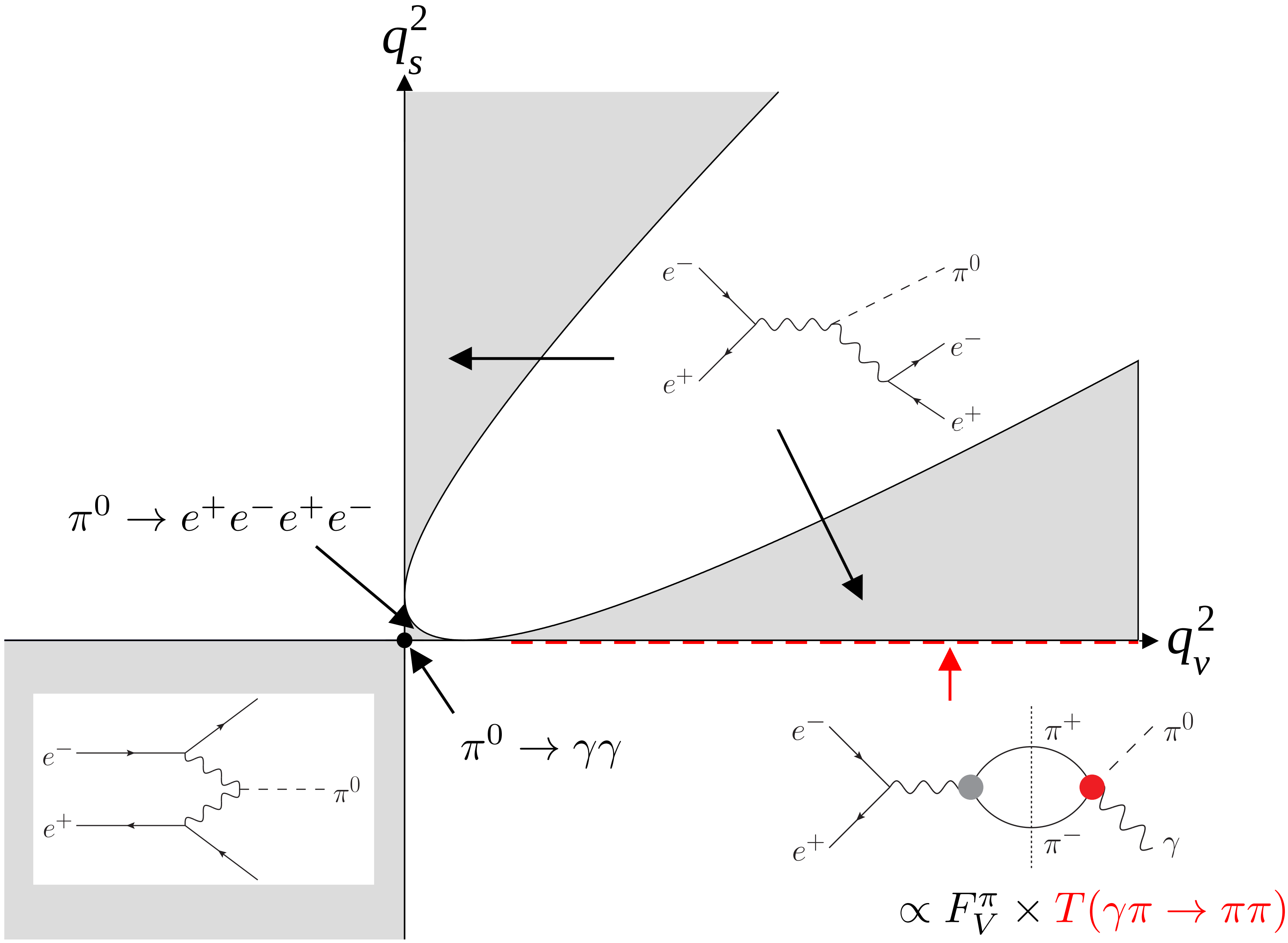

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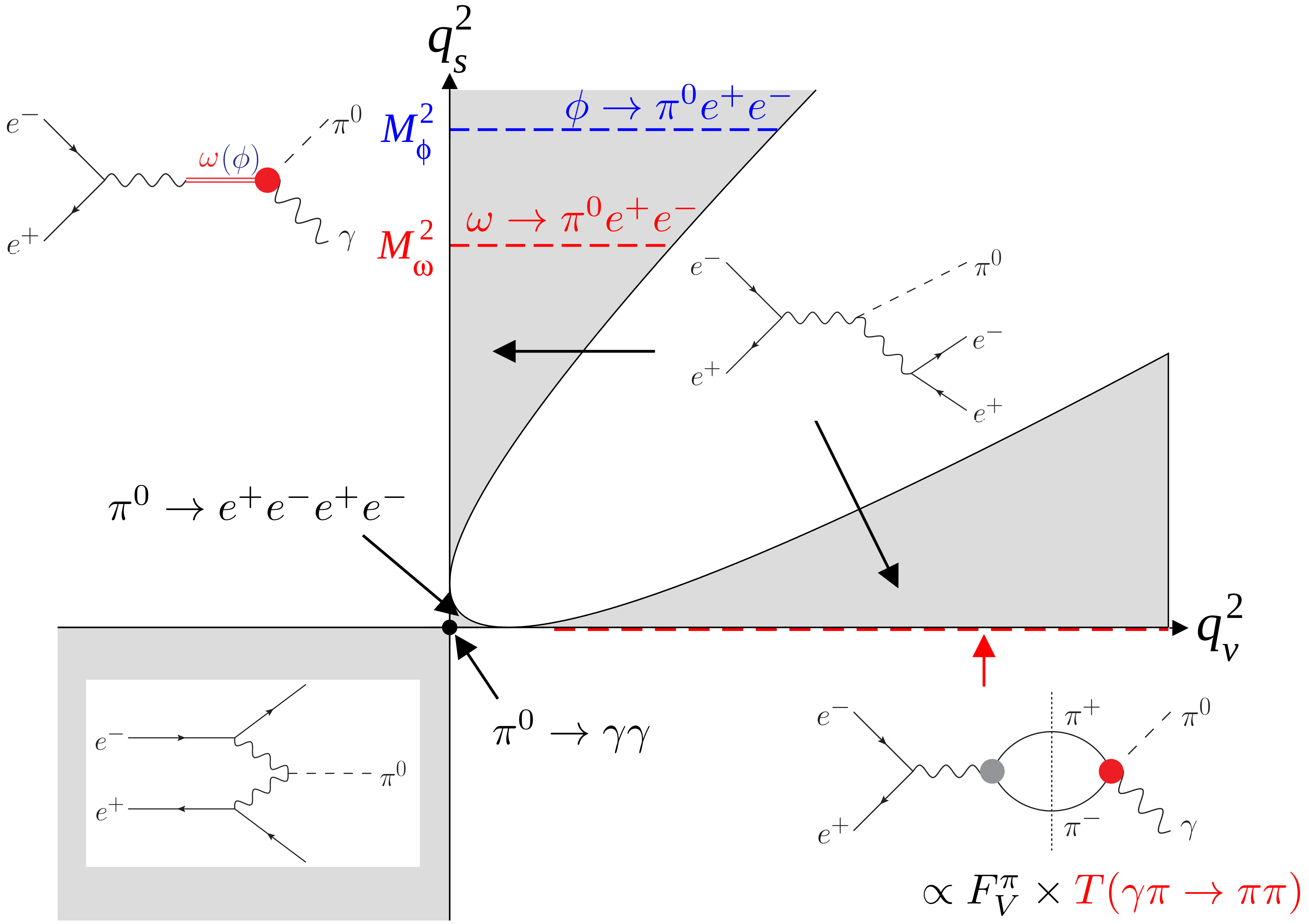

Anomalous process $\gamma\pi \to \pi\pi$

- $\gamma\pi \to \pi\pi$: crossing symmetry, WZW low-energy theorem

$$\mathcal{F}(s,t,u) = \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)\,, \qquad \mathcal{F}(0,0,0) = F_{3\pi} = \frac{e}{4\pi^2 F_\pi^3}$$

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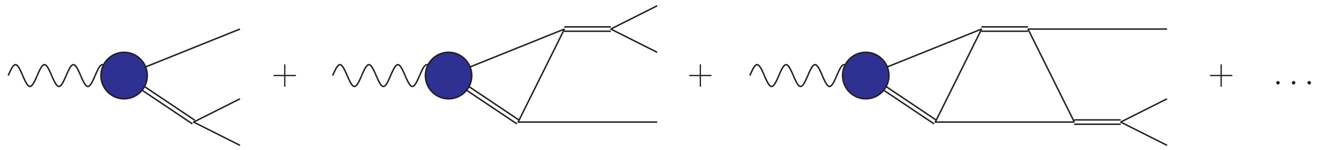
- left-hand cut $\hat{\mathcal{F}}(s)$ and right-hand cut $\mathcal{F}(s)$ self-consistent:

$$\mathcal{F}(s) = \Omega(s)\left\{\frac{C_1}{3} + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right\}$$

$=$ 

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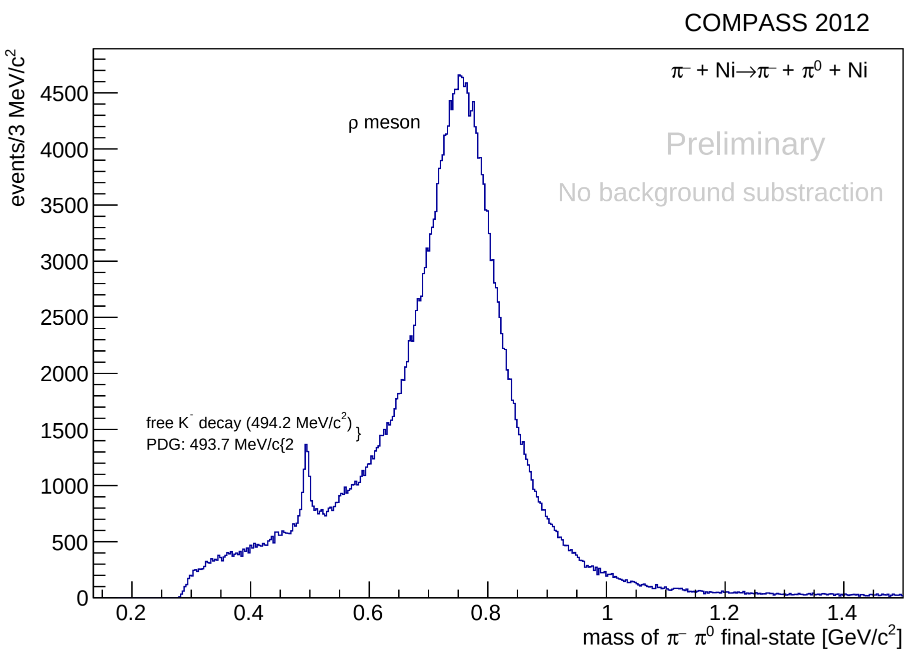
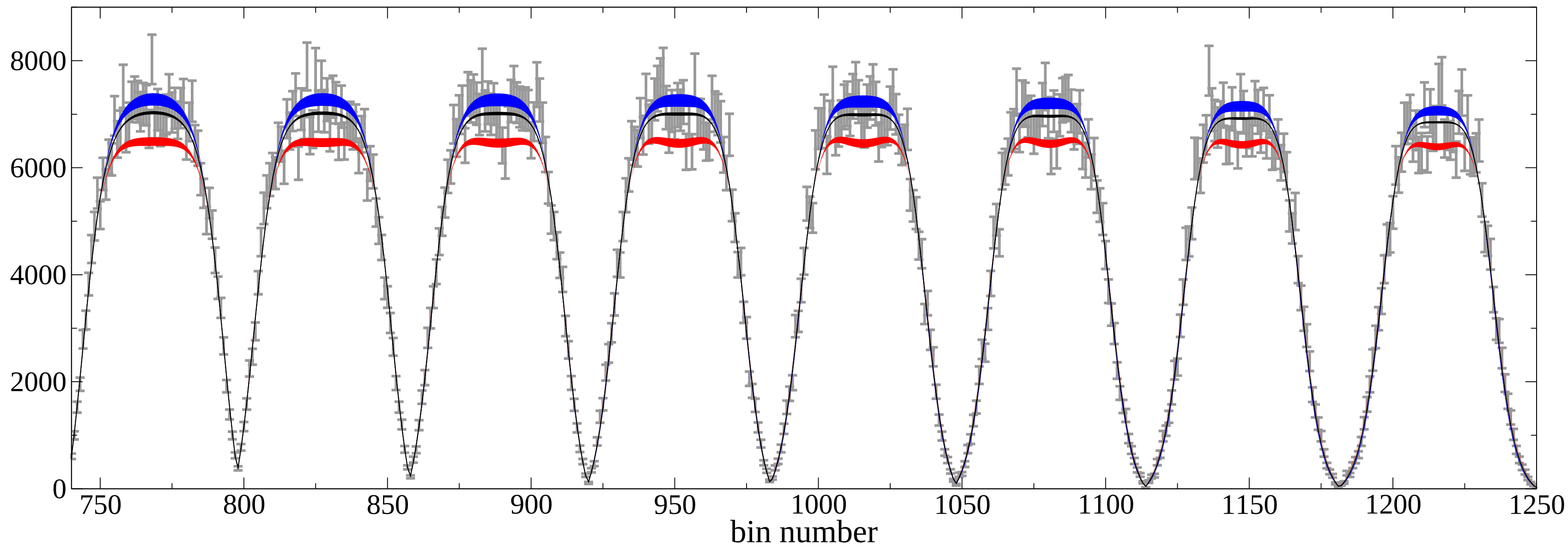
- high-accuracy data from Primakoff spectrum COMPASS
- fit dispersive representation to data, extract $F_{3\pi} \simeq C_1$ Hoferichter, BK, Sakkas 2012
- radiative couplings of ρ, ρ_3 Hoferichter, BK, Zanke 2017
- lattice HadSpec coll. 2015



Seyfried, MSc thesis 2017

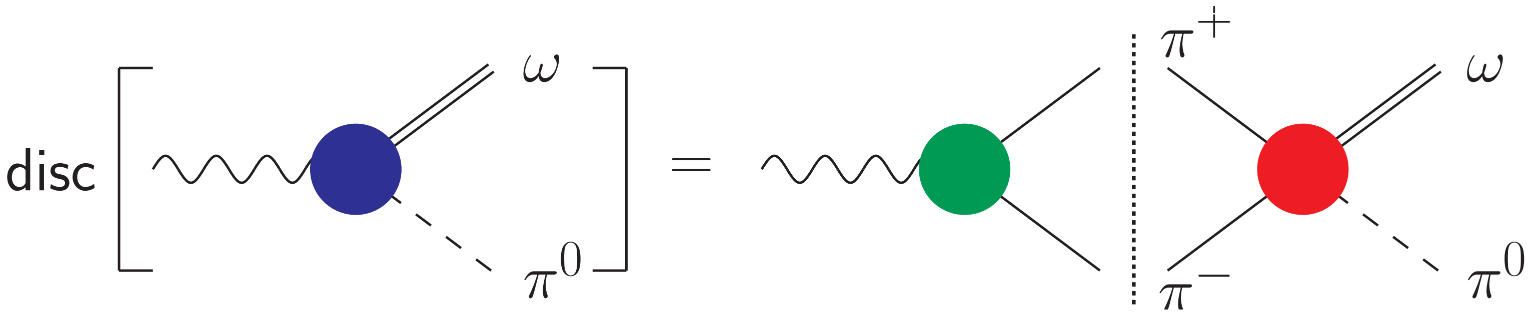
Extension to decays: $\omega/\phi \to 3\pi$

- same quantum numbers as $\gamma\pi \to \pi\pi$ — Niecknig, BK, Schneider 2012; Danilkin et al. 2015
- first $\omega \to 3\pi$ Dalitz plot measurement — WASA-at-COSY 2017
- test accuracy on $\phi \to 3\pi$ Dalitz plot: $2 \cdot 10^6$ events — KLOE 2005



	$\hat{\mathcal{F}} = 0$	once-subtracted	twice-subtracted
χ^2/ndof	$1.71 \ldots 2.06$	$1.17 \ldots 1.50$	$1.02 \ldots 1.03$

Transition form factor $\omega(\phi) \to \pi^0 \ell^+ \ell^-$



- ω transition form factor related to

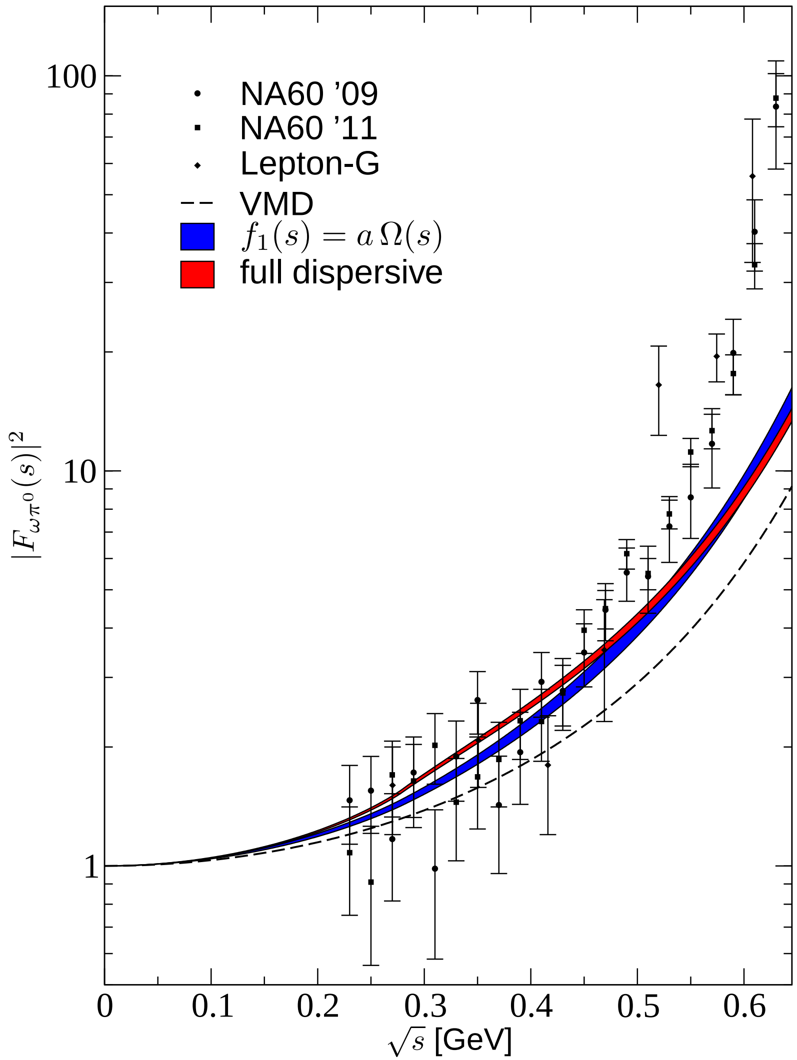
  pion vector form factor $\times$ $\omega \to 3\pi$ decay amplitude

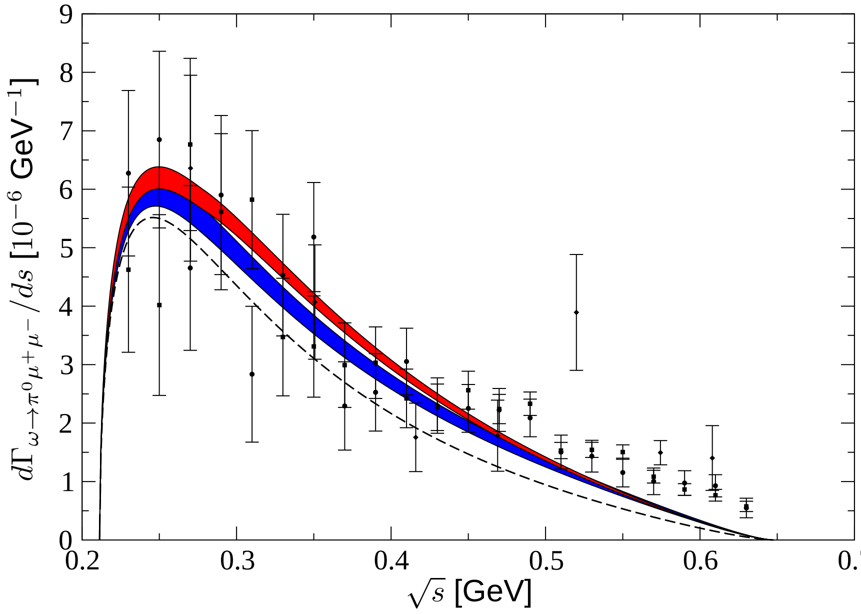
- form factor normalization yields rate $\Gamma(\omega \to \pi^0 \gamma)$

  (2nd most important ω decay channel)

  $\longrightarrow$ works at 95% accuracy Schneider, BK, Niecknig 2012

Numerical results: $\omega \to \pi^0\mu^+\mu^-$





- clear enhancement vs. VMD
- incompatible with data from heavy-ion collisions

  NA60 2009, 2011
- $\omega \to \pi^0 e^+ e^-$ data: no tension (but less precise) A2 2016
- NA60 data potentially in conflict with unitarity bounds

  Ananthanarayan, Caprini, BK 2014, Caprini 2015

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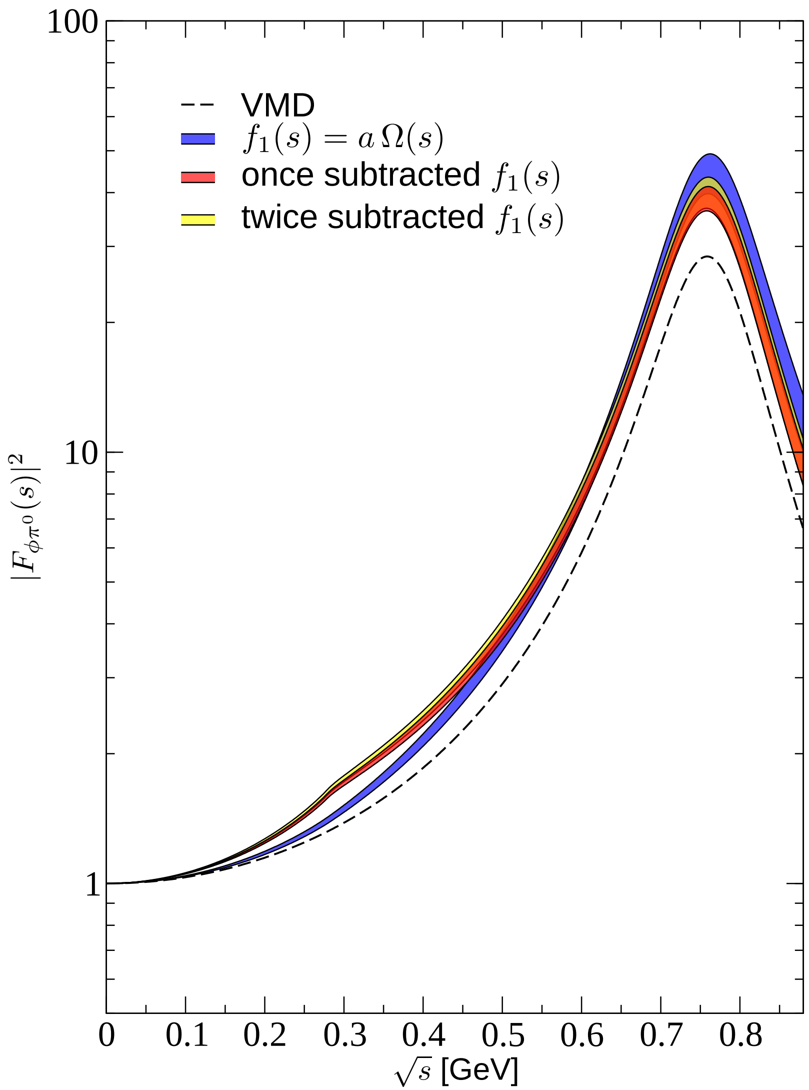
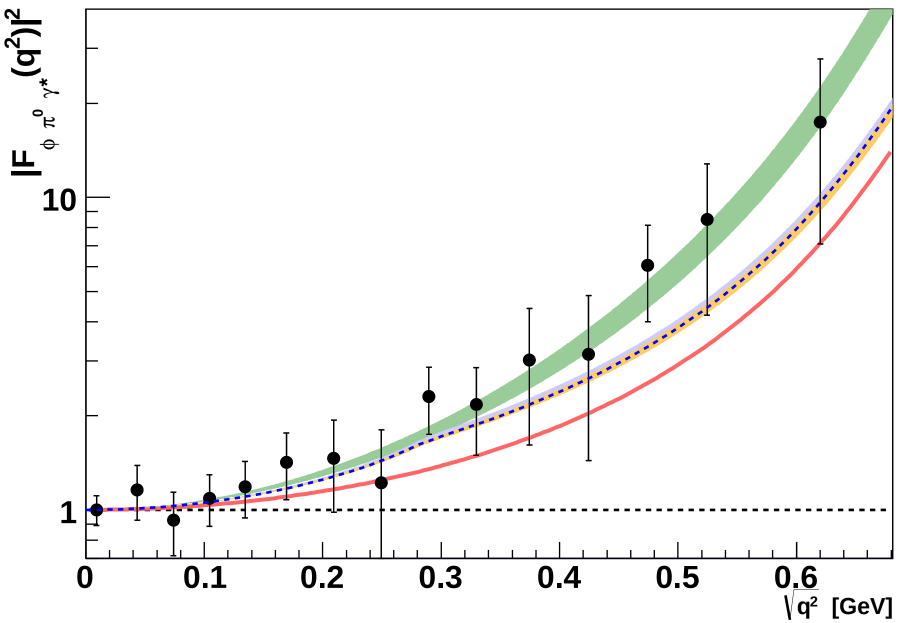


Schneider, BK, Niecknig 2012

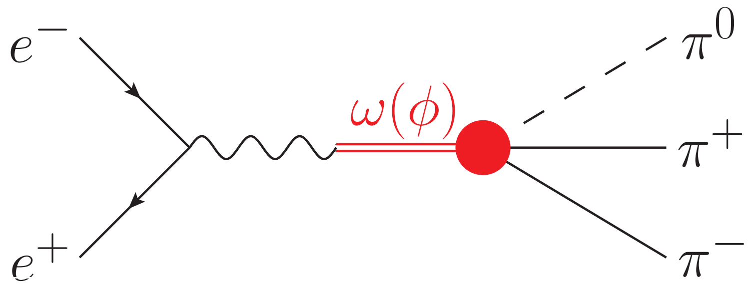


KLOE 2016

- measurement in ρ peak region would be extremely helpful
- $\phi \to 3\pi$ partial-wave amplitude backed up by experiment

Niecknig, BK, Schneider 2012

One step further: $e^+e^- \to 3\pi$, $e^+e^- \to \pi^0\gamma^*$



- decay amplitude for $\omega/\phi \to 3\pi$: $\mathcal{M}_{\omega/\phi} \propto \mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u)$

$$\mathcal{F}(s) = a_{\omega/\phi}\,\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right\}$$

$a_{\omega/\phi}$ adjusted to reproduce total width $\omega/\phi \to 3\pi$

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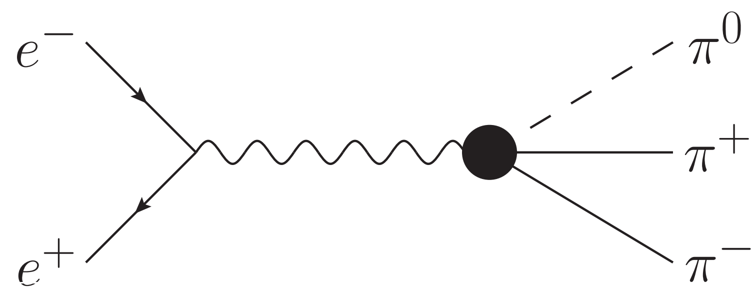

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$a_{e^+e^-}(q^2)$ adjusted to reproduce spectrum $e^+e^- \to 3\pi$

- parameterisation:

$$a_{e^+e^-}(q^2) = \frac{F_{3\pi}}{3} + \beta\, q^2 + \frac{q^4}{\pi}\int_{\text{thr}}^{\infty} ds' \frac{\text{Im}BW(s')}{s'^2(s'-q^2)}$$

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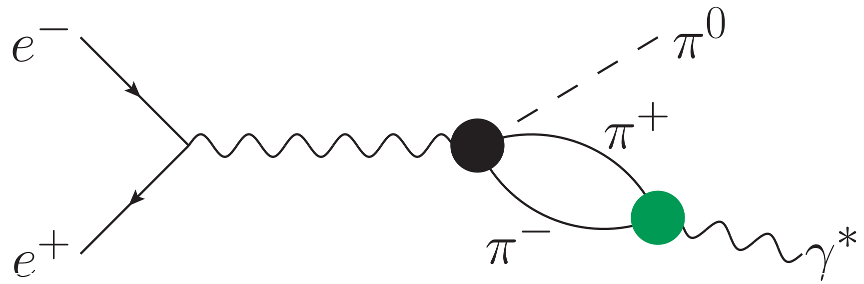

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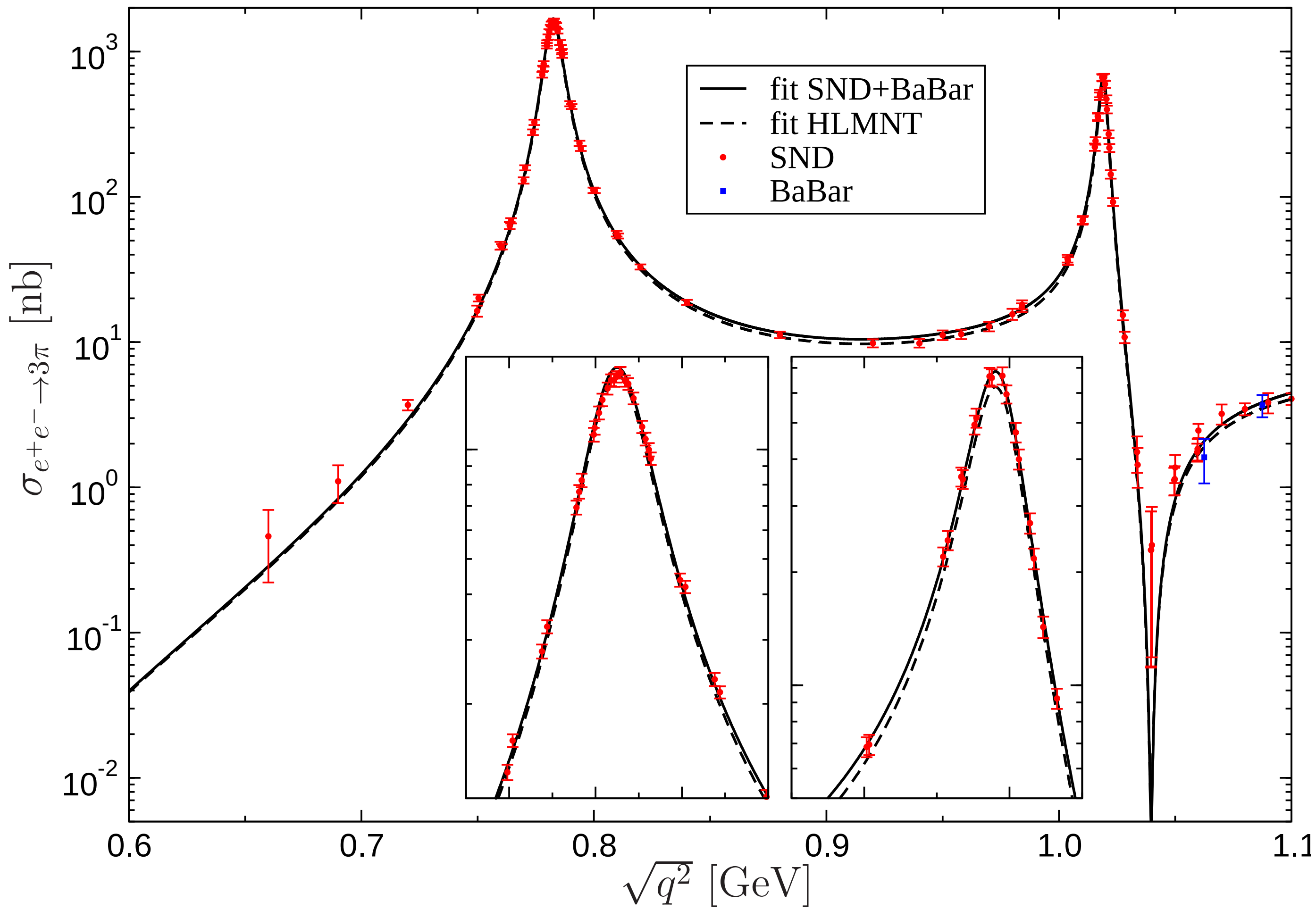
- fit to $e^+e^- \to 3\pi$ data $\longrightarrow$ prediction for $e^+e^- \to \pi^0\gamma^{(*)}$

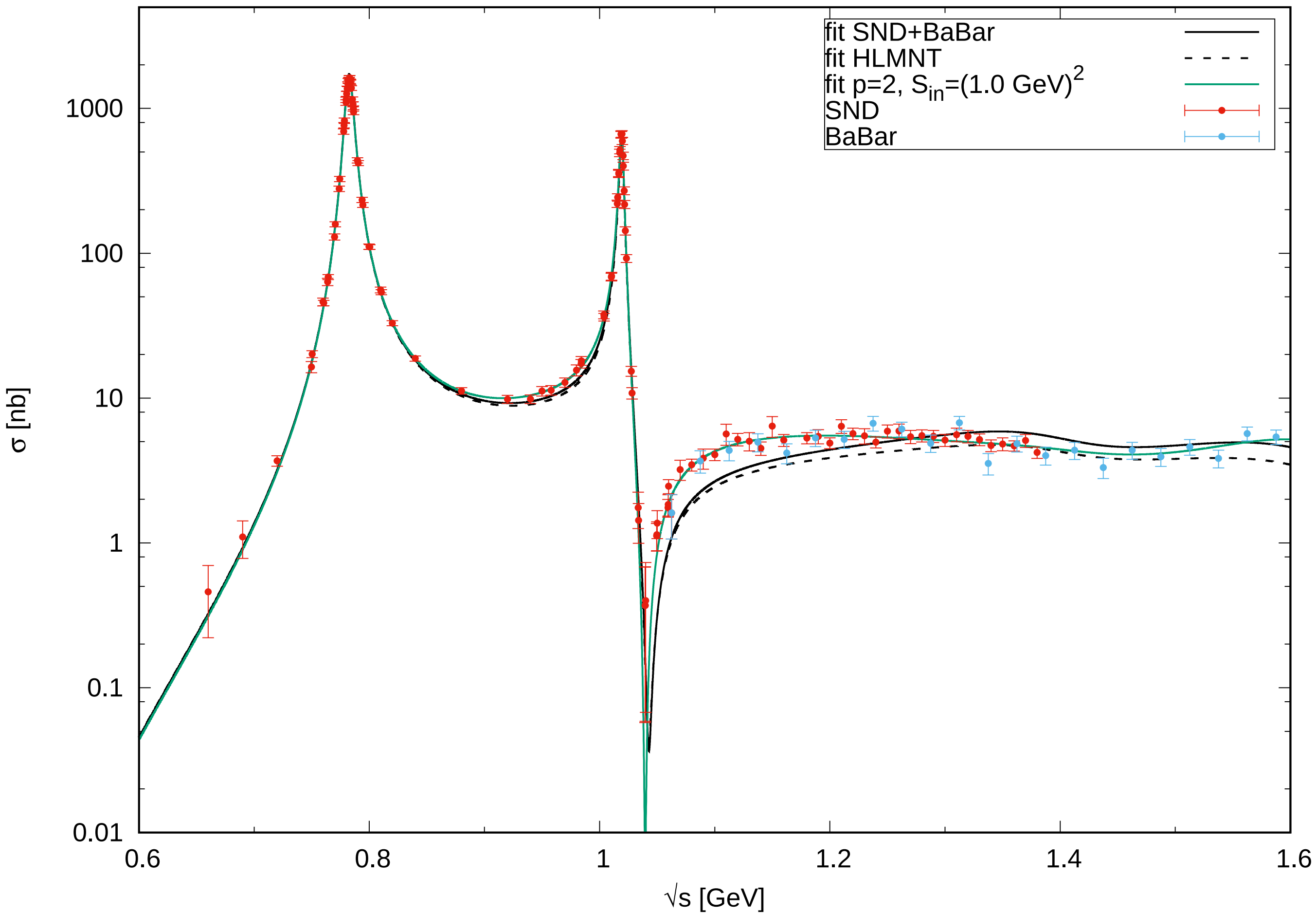
Fit to $e^+e^- \to 3\pi$ data



Hoferichter, BK, Leupold, Niecknig, Schneider 2014

- one subtraction/normalisation at $q^2 = 0$ fixed by $\gamma \to 3\pi$
- fitted: ω, ϕ residues, linear subtraction β

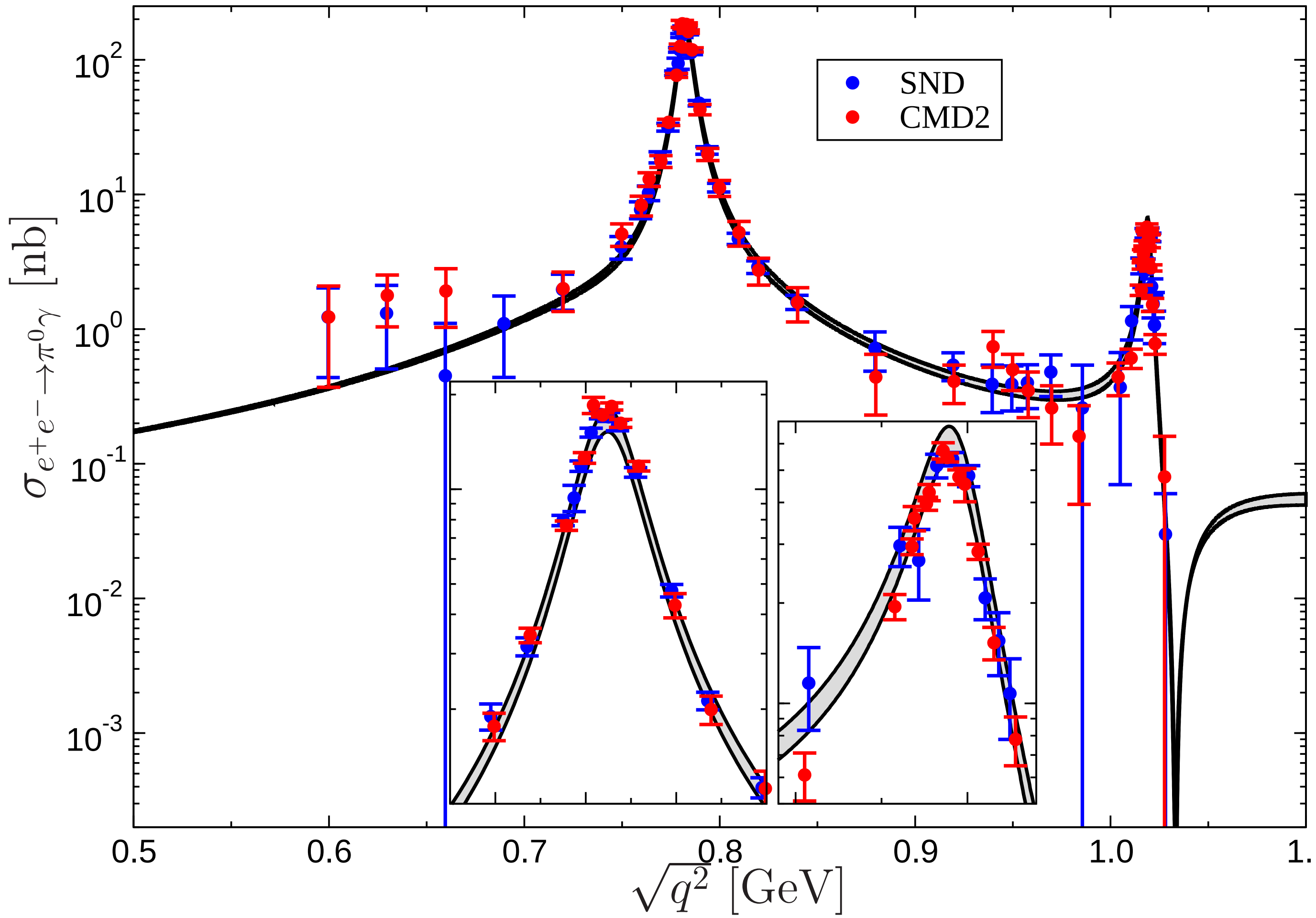
Improved fit to $e^+e^- \to 3\pi$ data



Long Bai, Hoferichter, BK, Leupold, Schneider, in progress

- improved description above the ϕ:
  unsubtracted DR, ω' poles, conformal subtraction polynomial
  $\longrightarrow$ exact implementation of $\gamma \to 3\pi$ anomaly (sum rule)

Comparison to $e^+e^- \to \pi^0\gamma$ data



Hoferichter, BK, Leupold, Niecknig, Schneider 2014

- "prediction"—no further parameters adjusted
- data very well reproduced

Extension to spacelike region; slope

- continuation to spacelike region: use another dispersion relation

$$F_{\pi^0\gamma^*\gamma}(q^2,0) = F_{\pi\gamma\gamma} + \frac{q^2}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\mathrm{Im}\, F_{\pi^0\gamma^*\gamma}(s',0)}{s'(s'-q^2)}$$

- sum rule for slope $\quad F_{\pi^0\gamma^*\gamma}(q^2,0) = F_{\pi\gamma\gamma}\Big\{1 + a_\pi \frac{q^2}{M_{\pi^0}^2} + \mathcal{O}(q^4)\Big\}$

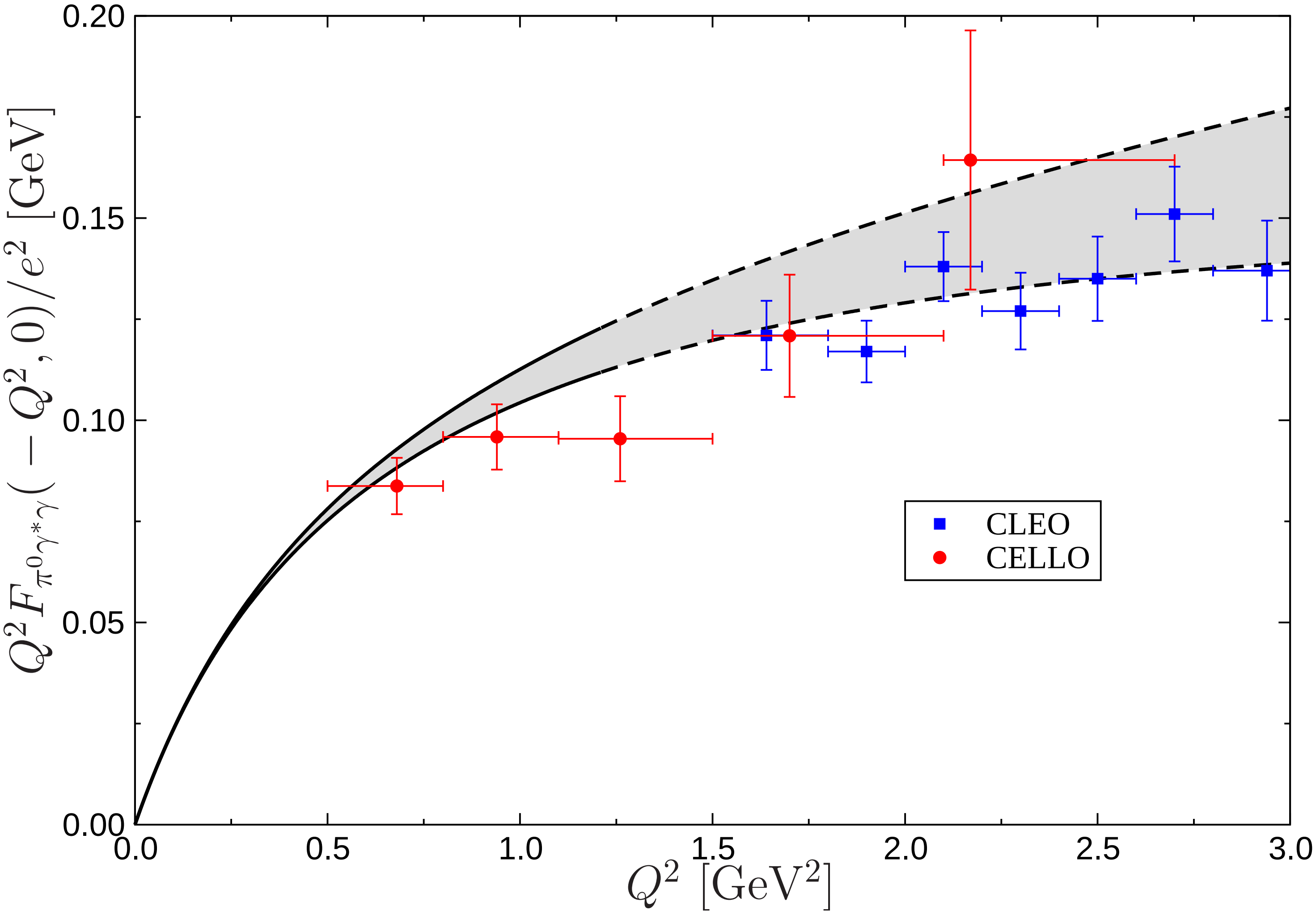
$$a_\pi = \frac{M_{\pi^0}^2}{F_{\pi\gamma\gamma}} \times \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^2}\,\mathrm{Im}\, F_{\pi^0\gamma^*\gamma}(s',0)$$

$$= (30.7 \pm 0.6) \times 10^{-3}$$ Hoferichter et al. 2014

compare: $\quad a_\pi = (32 \pm 4) \times 10^{-3}$ PDG 2014

- theory error estimate:
  - ▷ $\pi\pi$ phases
  - ▷ cutoff effects in $\gamma^* \to 3\pi$ partial waves and $[\gamma^* \to 3\pi] \longrightarrow [\gamma^* \to \pi^0\gamma]$

Prediction spacelike form factor



Hoferichter, BK, Leupold, Niecknig, Schneider 2014

$\longrightarrow$ more precise low-energy spacelike data to come BESIII

Towards the doubly-virtual π^0 transition form factor

- so far: subtracted dispersion relation
  - ▷ high precision at low energies
  - ▷ uses $\pi^0 \to \gamma\gamma$ as input for form factor normalisation
  - ▷ asymptotically: $\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma}\big(-Q^2,0\big) =$ const.

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,-Q^2\big) = \text{const.}$$

$\longrightarrow$ unfit for $(g-2)_\mu$ implementation

$\longrightarrow$ switch to unsubtracted representation

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  - ▷ high precision at low energies
  - ▷ uses $\pi^0 \to \gamma\gamma$ as input for form factor normalisation
  - ▷ asymptotically: $\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma}\big(-Q^2,0\big) =$ const.

$$\lim_{Q^2\to\infty} F_{\pi^0\gamma^*\gamma^*}\big(-Q^2,-Q^2\big) = \text{const.}$$

$\longrightarrow$ unfit for $(g-2)_\mu$ implementation

$\longrightarrow$ switch to unsubtracted representation

- double-spectral-function representation:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_{4M_\pi^2}^{\infty} dx \int_{s_{\rm thr}}^{\infty} dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}$$

$$\rho(x,y) = \frac{q_\pi^3(x)}{12\pi\sqrt{x}} {\rm Im}\Big[F_\pi^{V*}(x) f_1(x,y)\Big] + [x\leftrightarrow y]$$

$f_1(s,q^2)$: $\gamma^*(q^2)\pi \to \pi\pi$ P-wave

$\longrightarrow$ doubly-virtual form factor fixed from singly-virtual input!

Asymptotics and pQCD constraints (1)

- asymptotically, π^0 TFF can be expressed via pion wave function:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = -\frac{2e^2 F_\pi}{3}\int_0^1 dx \frac{\phi_\pi(x)}{xq_1^2 + (1-x)q_2^2} + \mathcal{O}\big(Q^{-4}\big)$$

$$\phi_\pi(x) = 6x(1-x) + \dots$$

Brodsky, Lepage 1979–1981

- implies

$$F_{\pi^0\gamma^*\gamma^*}\big(-Q^2, -Q^2\big) = \frac{2e^2 F_\pi}{3Q^2} + \mathcal{O}\big(Q^{-4}\big)$$

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- change of variables: rewrite this as dispersion relation

$$F^{\text{pQCD}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi}\int_0^\infty dx \frac{\operatorname{Im} F^{\text{pQCD}}_{\pi^0\gamma^*\gamma^*}(x,q_2^2)}{x-q_1^2}$$

$$\operatorname{Im} F^{\text{pQCD}}_{\pi^0\gamma^*\gamma^*}(x,y) = \frac{2\pi e^2F_\pi}{3(x-y)}\,\phi_\pi\Big(\frac{x}{x-y}\Big)$$

Khodjamirian 1999

Asymptotics and pQCD constraints (2)

- formally, write this as a double-spectral representation:

$$F^{\mathrm{pQCD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_0^\infty dx \int_0^\infty dy \frac{\rho^{\mathrm{pQCD}}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

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- decomposition of the full form factor:

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{1}{\pi^2}\int_0^{s_\mathrm{m}} dx \int_0^{s_\mathrm{m}} dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_\mathrm{m}}^\infty dx \int_{s_\mathrm{m}}^\infty dy \frac{\rho(x,y)}{(x-q_1^2)(y-q_2^2)}$$
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- use dispersive $\rho(x,y)$ at low energies
- doubly-asymptotic: $\rho^{\mathrm{pQCD}}(x,y) \propto x\,y$

  $\longrightarrow$ does not contribute to singly-virtual TFF!
- mixed regions: nothing known, $\rho^{\mathrm{pQCD}}(x,y)$ vanishes there

  $\longrightarrow$ all constraints can be fulfilled setting these to zero

Asymptotics and pQCD constraints (3)

$$F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{1}{\pi^2}\int_0^{s_\mathrm{m}} dx \int_0^{s_\mathrm{m}} dy \frac{\rho^{\mathrm{disp}}(x,y)}{(x-q_1^2)(y-q_2^2)} + \frac{1}{\pi^2}\int_{s_\mathrm{m}}^{\infty} dx \int_{s_\mathrm{m}}^{\infty} dy \frac{\rho^{\mathrm{pQCD}}(x,y)}{(x-q_1^2)(y-q_2^2)}$$

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- dispersive part: $\dfrac{1}{\pi^2}\displaystyle\int_0^{s_\mathrm{m}} dx \int_0^{s_\mathrm{m}} dy \frac{\rho^{\mathrm{disp}}(x,y)}{(x+Q^2)(y+Q^2)} = \mathcal{O}\big(Q^{-4}\big)$

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$$\frac{1}{\pi^2}\int_0^{s_\mathrm{m}} dx \int_0^{s_\mathrm{m}} dy \frac{\rho^\mathrm{disp}(x,y)}{x\,y} = \frac{e^2}{4\pi^2F_\pi} \qquad \text{[anomaly]}$$

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- add effective pole: $\rho^\mathrm{eff} = \frac{e^2 g_\mathrm{eff}}{4\pi^2F_\pi}\pi^2M_\mathrm{eff}^4\delta\big(x-M_\mathrm{eff}^2\big)\delta\big(y-M_\mathrm{eff}^2\big)$

  find $g_\mathrm{eff} \sim 3\ldots 8\%$ (small), $M_\mathrm{eff} \sim 1.8\ldots 2.5\,\mathrm{GeV}$ (reasonable)

Uncertainties in the dispersive approach

Normalisation

- uncertainty on $\pi^0 \to \gamma\gamma$ $\pm 0.85\%$ ($\longrightarrow$ down to $\pm 0.7\%$) PrimEx prel.
- implemented via effective pole coupling g_{eff}

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Dispersive input

- different phase shift inputs:
  - ▷ Bern phase — Colangelo, Caprini, Leutwyler
  - ▷ Madrid phase — García-Martín et al. 2011
  - ▷ effective form factor phase (incl. ρ', ρ'') — Schneider et al. 2012
- cutoff in Khuri–Treiman integrals $1.8 \ldots 2.5\,\text{GeV}$

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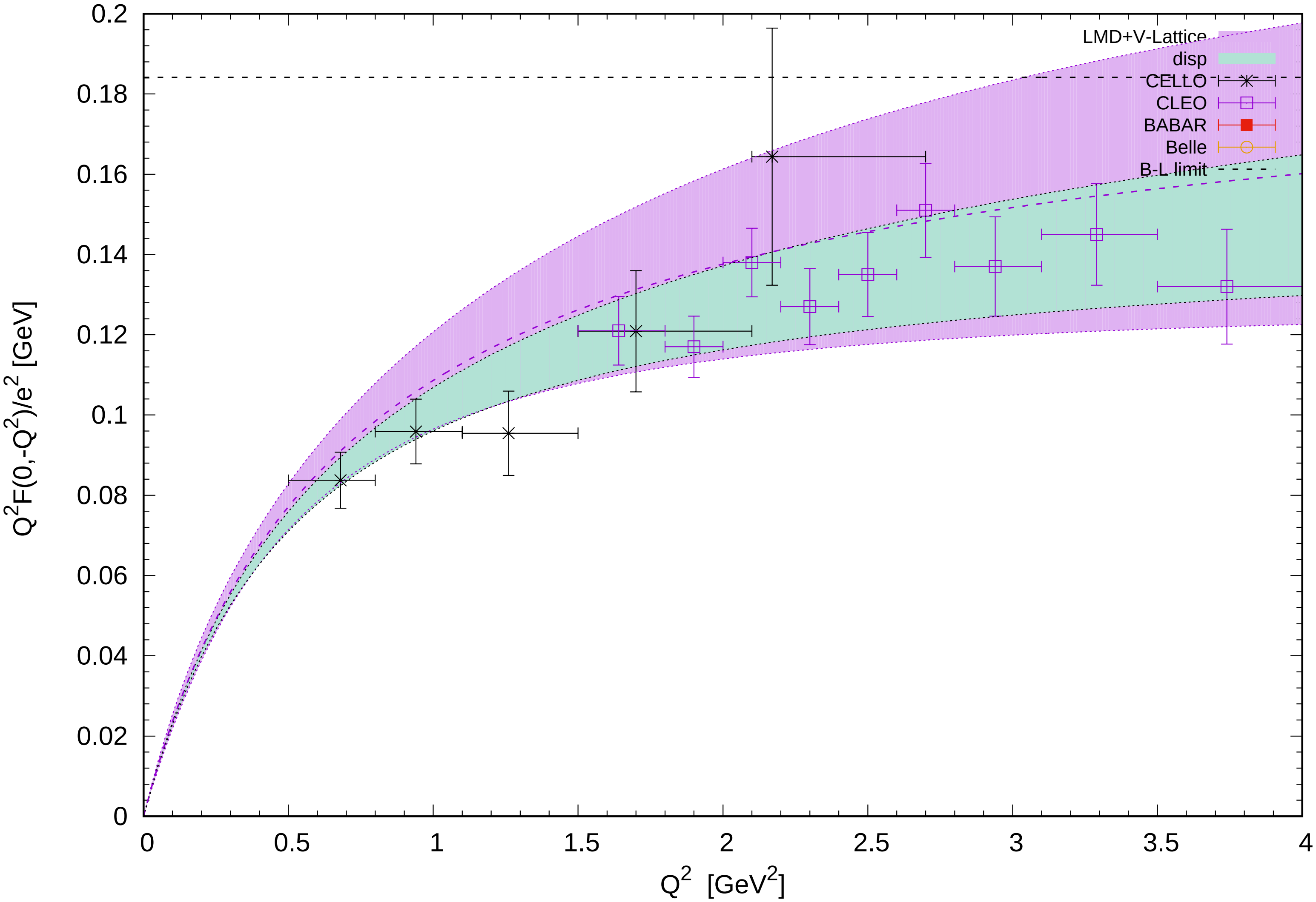
Effective pole

- variation in effective pole mass M_{eff}
  $\longrightarrow$ uncertainty around Brodsky–Lepage limit, here $\pm 25\%$

Preliminary results: singly-virtual

in comparison to Gérardin, Meyer, Nyffeler 2016



Long Bai, Hoferichter, BK, Leupold, Schneider, in progress

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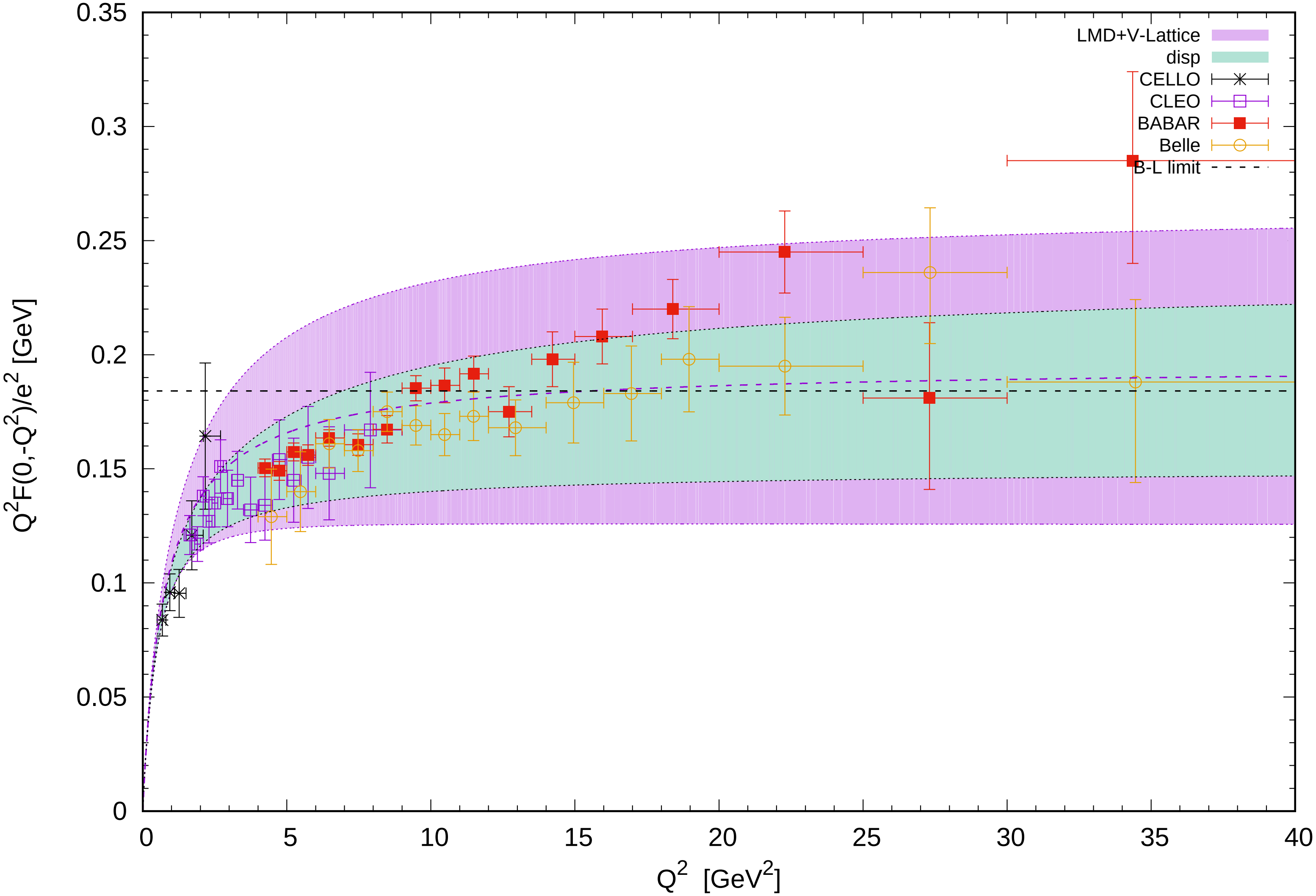

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Preliminary results: doubly-virtual (diagonal)

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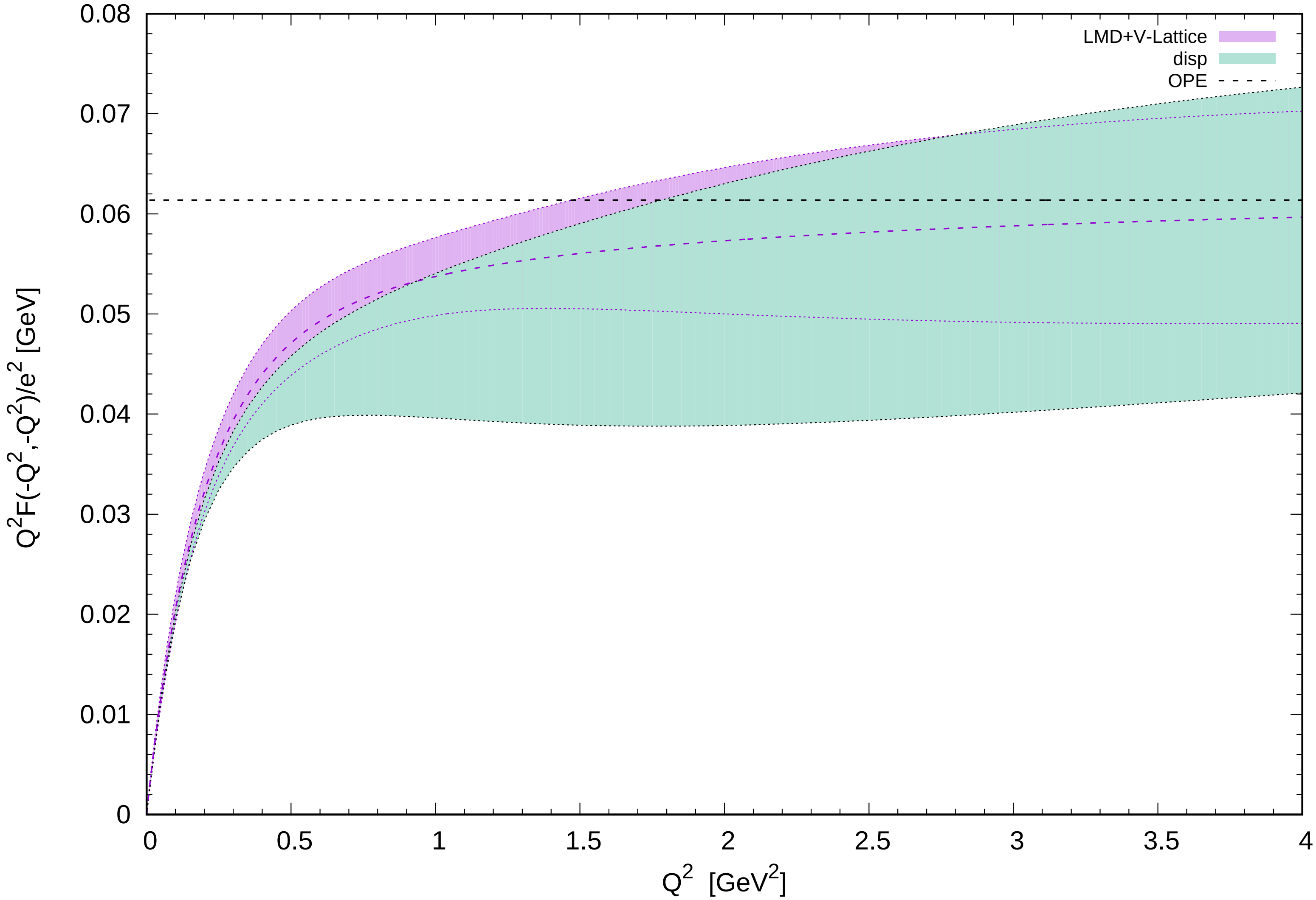

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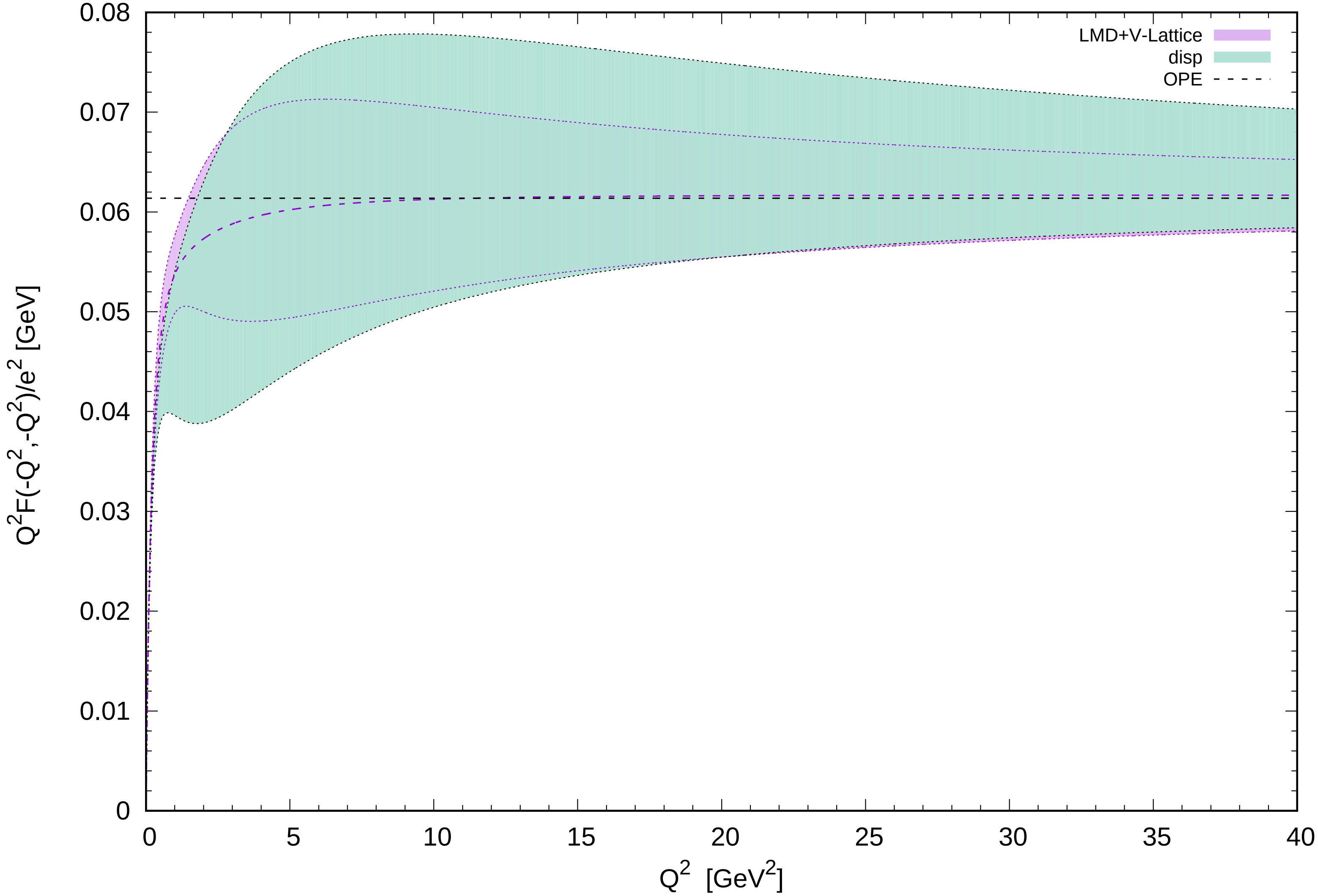
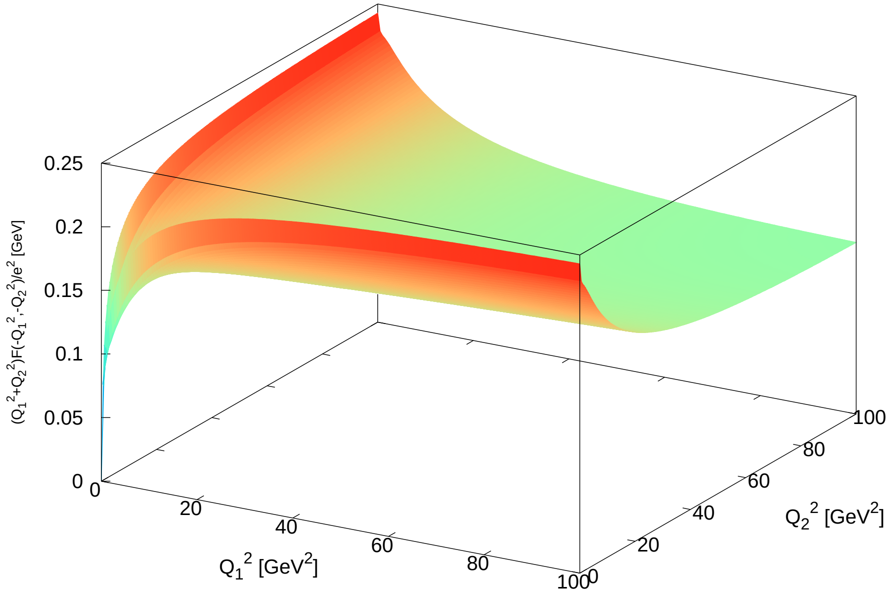


Long Bai, Hoferichter, BK, Leupold, Schneider, in progress

Comparison dispersive vs. LMD+V-lattice

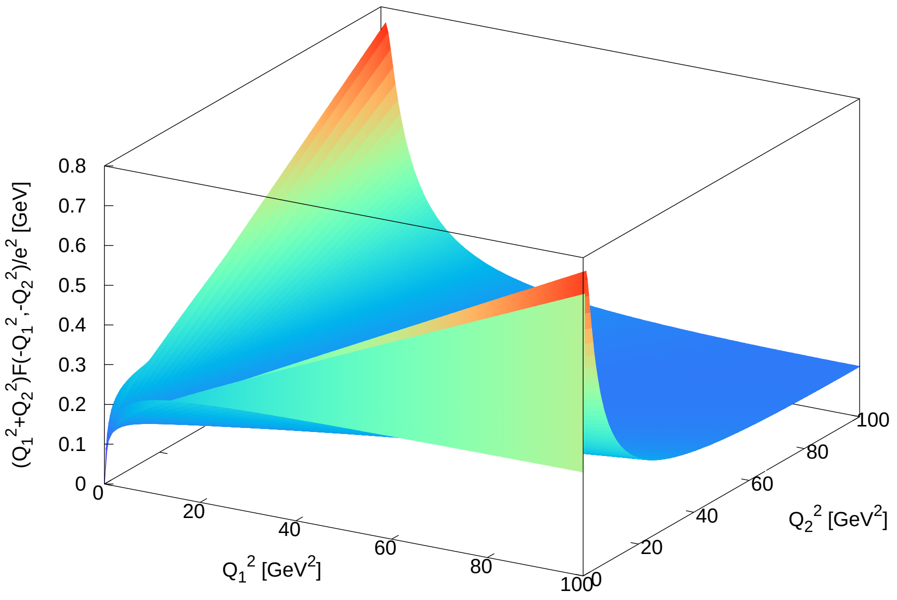
- plot $\left(Q_1^2 + Q_2^2\right) F_{\pi^0\gamma^*\gamma^*}\left(-Q_1^2, -Q_2^2\right)$:

dispersive



Long Bai, Hoferichter, BK,
Leupold, Schneider, in progress

LMD+V fit to lattice



Gérardin, Meyer, Nyffeler 2016

Preliminary results: $(g-2)_\mu$ from π^0 pole

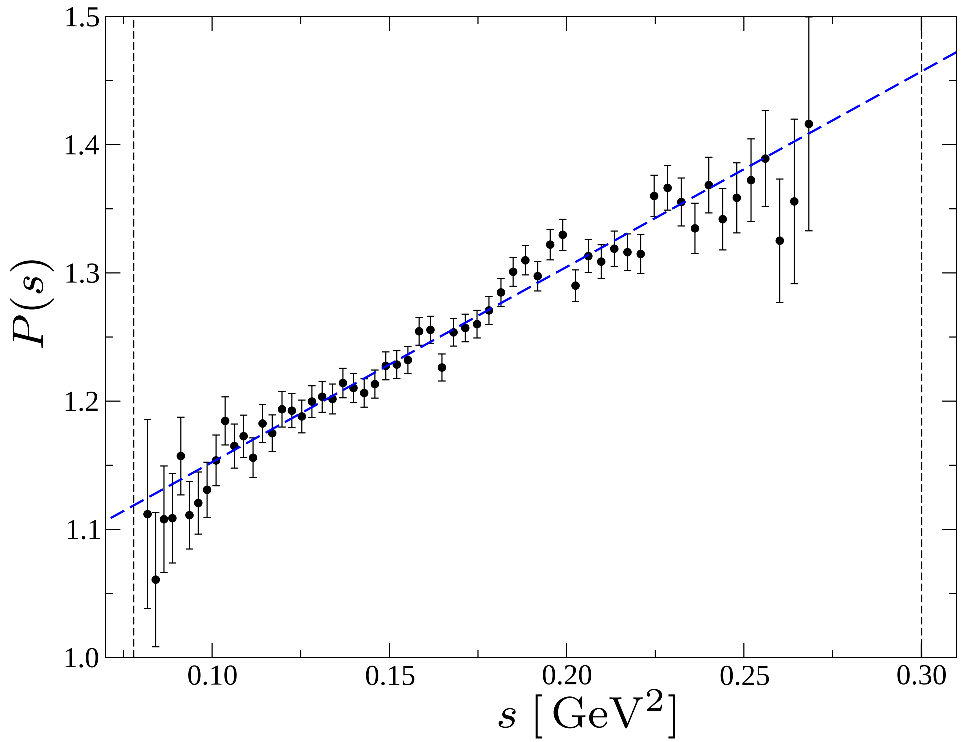
Very preliminary estimate of the π^0 pole contribution $[10^{-11}]$

60.2 ± 1.4	chiral anomaly / $\pi^0 \to \gamma\gamma$
± 1.6	dispersive input
${}^{+\,1.4}_{-\,1.8}$	Brodsky–Lepage ${}^{+25\%}_{-20\%}$
± 0.4	onset of pQCD contribution s_m

- dominant uncertainties are all accessible in real-photon or singly-virtual experiments

Final-state universality: $\eta,\ \eta' \to \pi^+\pi^-\gamma$

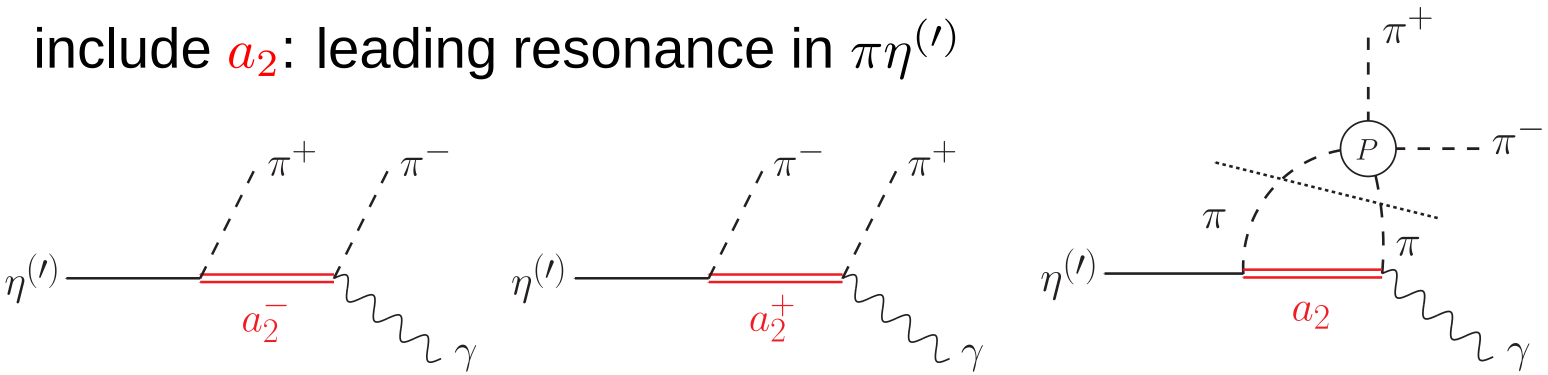
- $\eta^{(\prime)} \to \pi^+\pi^-\gamma$ driven by the chiral anomaly, $\pi^+\pi^-$ in P-wave
  $\longrightarrow$ final-state interactions the same as for vector form factor
- ansatz: $\mathcal{F}^{\eta^{(\prime)}}_{\pi\pi\gamma} = A \times P(s) \times \Omega(s),\ \ P(s) = 1 + \alpha^{(\prime)} s,\ \ s = M^2_{\pi\pi}$
- divide data by pion form factor $\longrightarrow P(s)$ — Stollenwerk et al. 2012

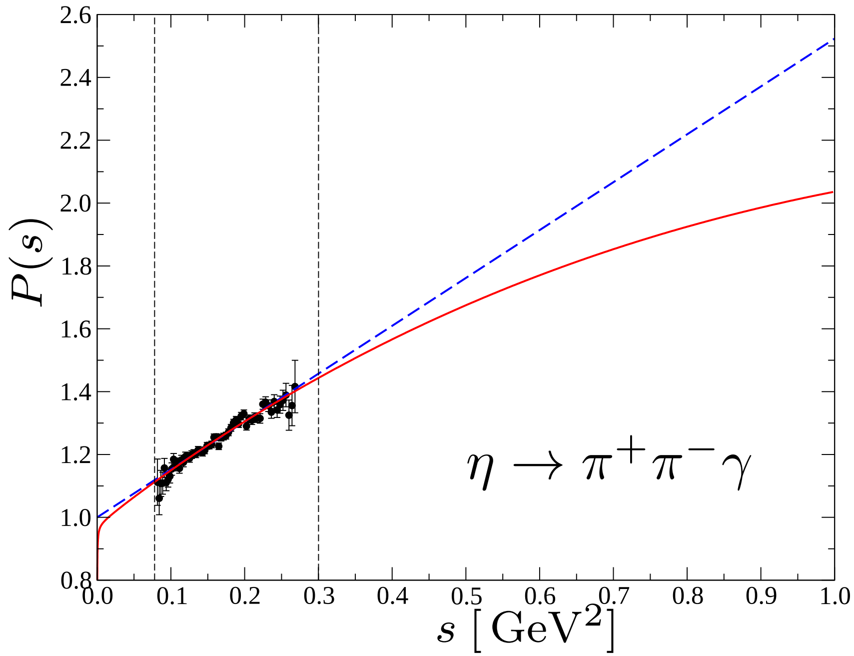


$\longrightarrow$ exp.: $\alpha_{\text{KLOE}} = (1.52 \pm 0.06)$ GeV^{-2} — cf. KLOE 2013

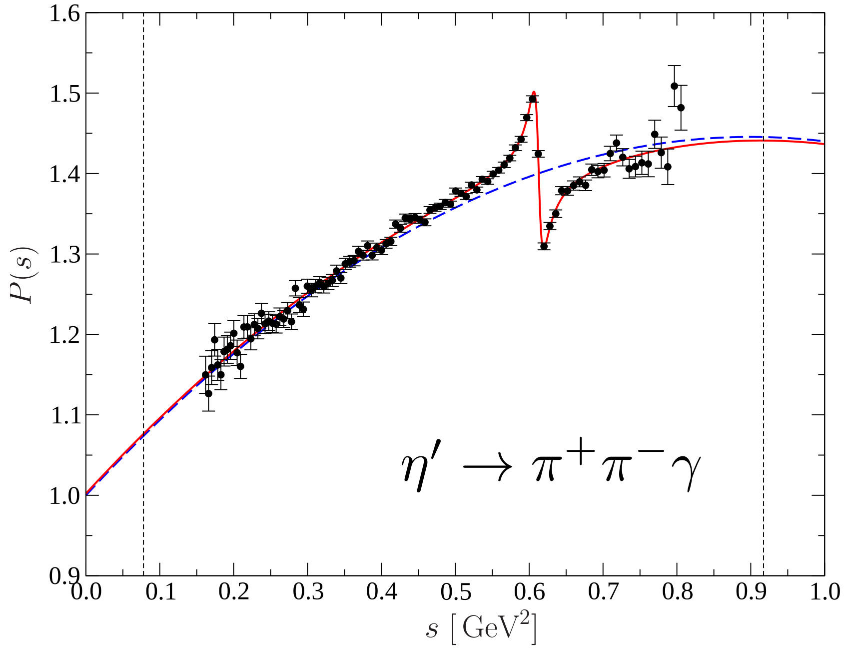
$\eta,\ \eta' \to \pi^+\pi^-\gamma$ with left-hand cuts

- include a_2: leading resonance in $\pi\eta^{(\prime)}$





KLOE 2013; BK, Plenter 2015



BESIII 2017; Hanhart et al. 2017

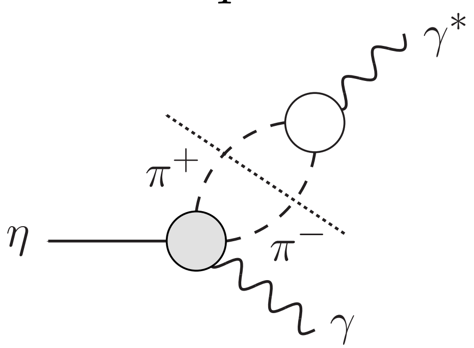
- induces curvature in $P(s)$
- curvature, plus ρ–ω mixing

Transition form factor $\eta \to \gamma^*\gamma$

Hanhart et al. 2013, BK, Plenter 2015

$$\bar{F}_{\eta\gamma^*\gamma}(q^2,0) = 1 + \frac{\kappa_\eta q^2}{96\pi^2 F_\pi^2}\int_{4M_\pi^2}^{\infty} ds\sigma(s)^3 P_V(s) P_\eta(s) \frac{|\Omega(s)|^2}{s-q^2}$$

$$+ \Delta F^{I=0}_{\eta\gamma^*\gamma}(q^2,0) \; [\longrightarrow \text{VMD}]$$



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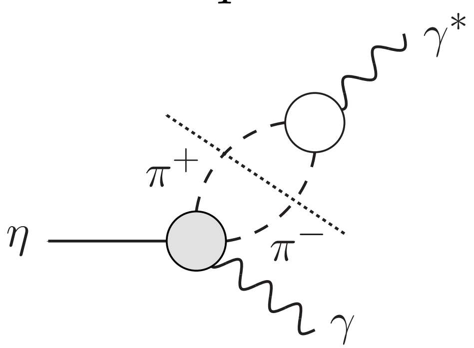
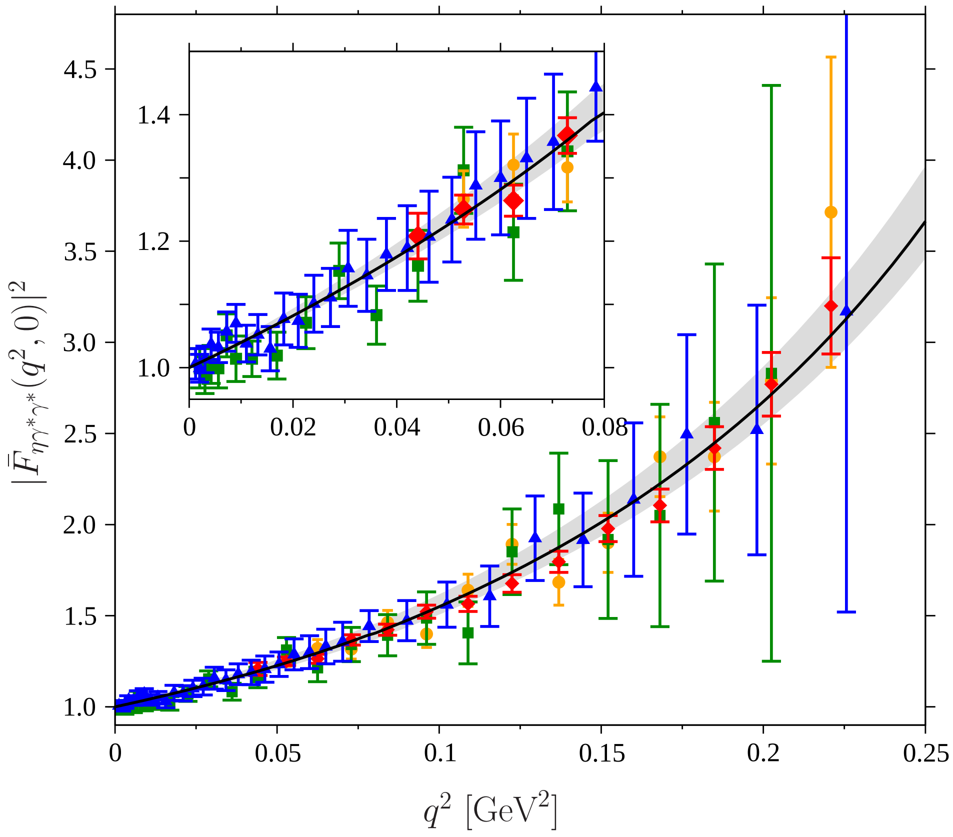


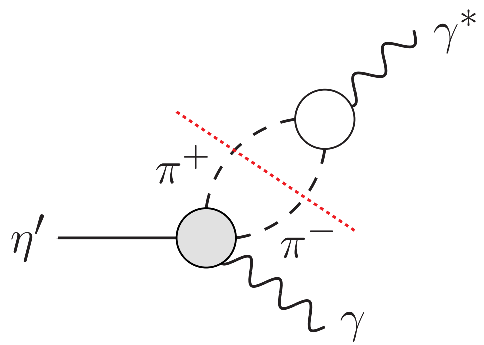


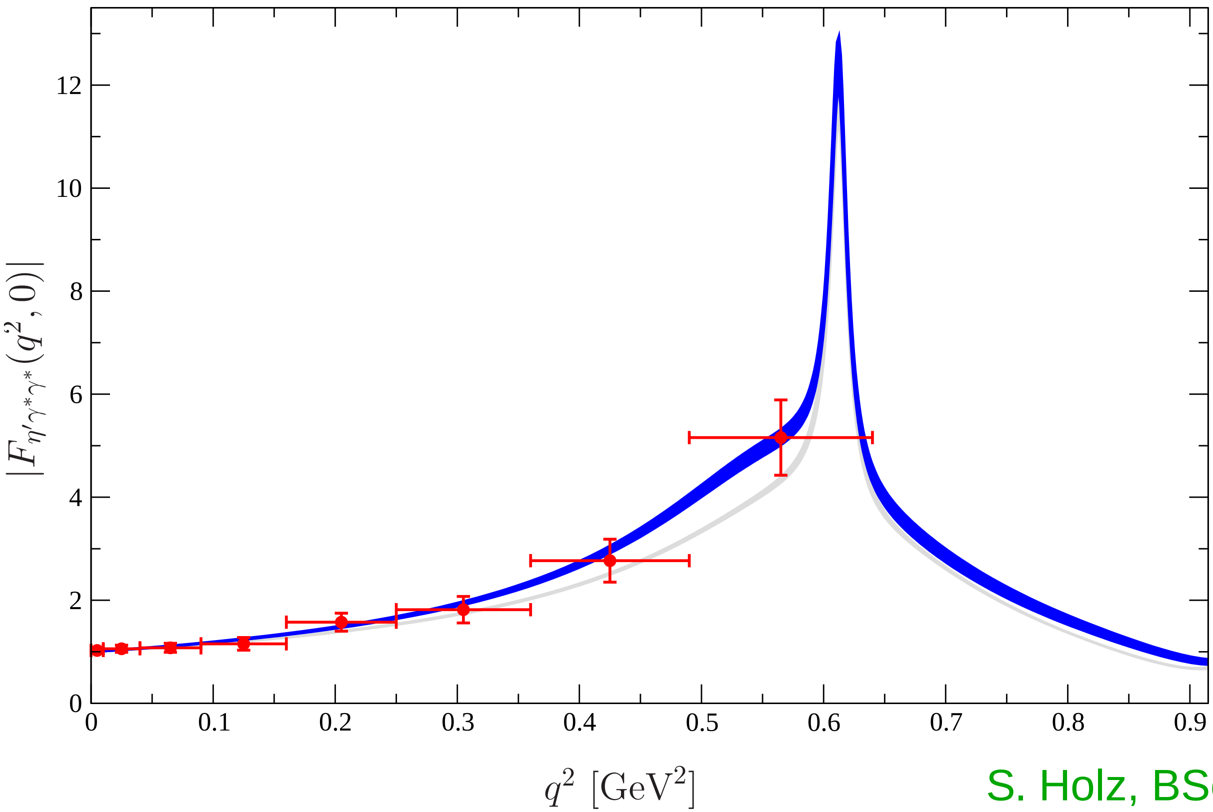
$\longrightarrow$ huge statistical advantage of using hadronic input for $\eta \to \pi^+\pi^-\gamma$ over direct measurement of $\eta \to \ell^+\ell^-\gamma$ (rate suppressed by α^2_{QED})

data: NA60 2009, 2016
A2 2014, 2017

Prediction for η' transition form factor

- isovector: combine high-precision data on $\eta' \to \pi^+\pi^-\gamma$ and $e^+e^- \to \pi^+\pi^-$
- isoscalar: VMD, couplings fixed from $\eta' \to \omega\gamma$ and $\phi \to \eta'\gamma$

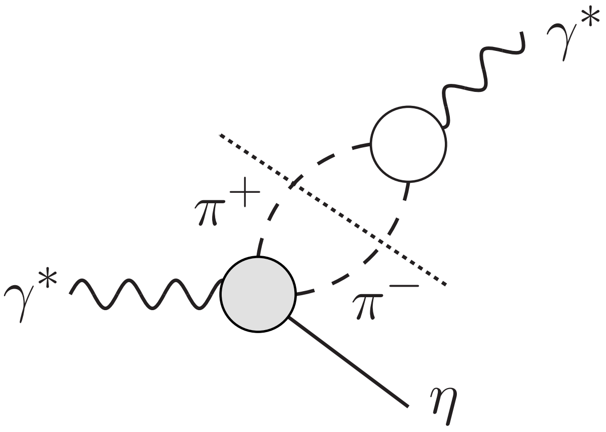




S. Holz, BSc thesis 2016
data: BESIII 2015

How to go *doubly* virtual? — $e^+e^- \to \eta\pi^+\pi^-$

- idea (again): beat α^2_{QED} suppression of $e^+e^- \to \eta e^+e^-$ by measuring $e^+e^- \to \eta\pi^+\pi^-$ instead



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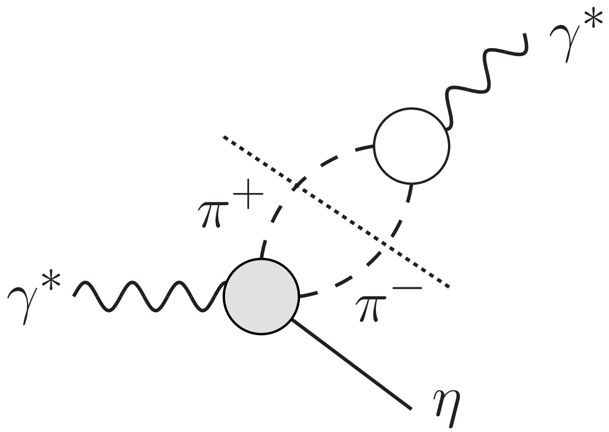

- test factorisation hypothesis in $e^+e^- \to \eta\pi^+\pi^-$:

$$f_1^{\eta\pi\pi\gamma^*}(s,k^2) \overset{!?}{=} f_1^{\eta\pi\pi\gamma}(s) \times F^{\eta\gamma\gamma^*}(k^2)$$

  - ▷ allow same form for $f_1^{\eta\pi\pi\gamma}(s)$ as in $\eta \to \pi^+\pi^-\gamma$
  - ▷ fit subtractions to $\pi^+\pi^-$ distribution in $e^+e^- \to \eta\pi^+\pi^-$
    $\longrightarrow$ are they compatible to the ones in $\eta \to \pi^+\pi^-\gamma$?
  - ▷ parametrise $F^{\eta\gamma\gamma^*}(k^2)$ by sum of Breit–Wigners (ρ, ρ')

Xiao et al. (preliminary)

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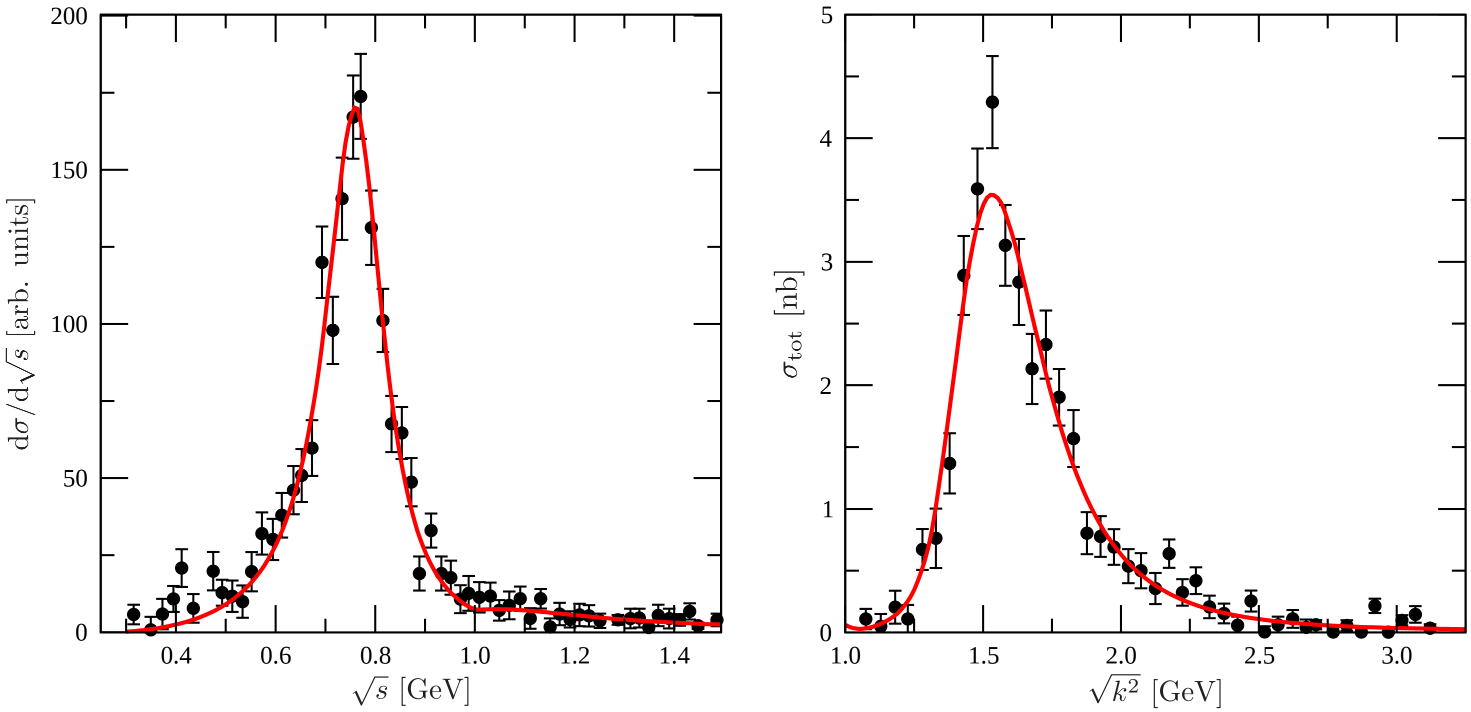


$$\frac{d\sigma}{d\sqrt{s}}$$

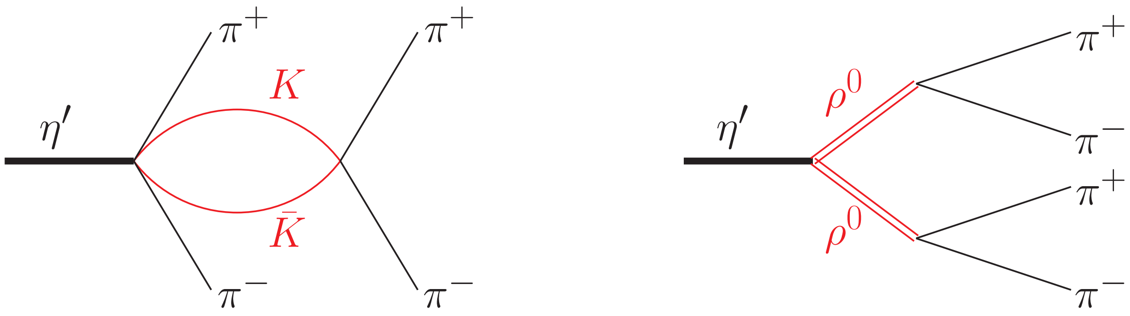
$$\sigma_{\text{tot}}(k^2)$$

Xiao et al. (preliminary); data: BaBar 2007

- $d\sigma/d\sqrt{s}$ integrated over $1\,\text{GeV} \leq \sqrt{k^2} \leq 4.5\,\text{GeV}$
- factorisation seems to work only if curvature ($\simeq a_2$) retained
- more differential/binned data highly desirable!

How to go *doubly* virtual? — $\eta' \to \pi^+\pi^-\pi^+\pi^-$

- prediction of $\eta' \to 4\pi$ branching ratios based on ChPT + VMD:



$\longrightarrow \quad \mathcal{B}(\eta' \to \pi^+\pi^-\pi^+\pi^-) = (10 \pm 3) \times 10^{-5}$ Guo, BK, Wirzba 2012

exp: $\quad \mathcal{B}(\eta' \to \pi^+\pi^-\pi^+\pi^-) = (8.5 \pm 0.7 \pm 0.6) \times 10^{-5}$ BESIII 2014

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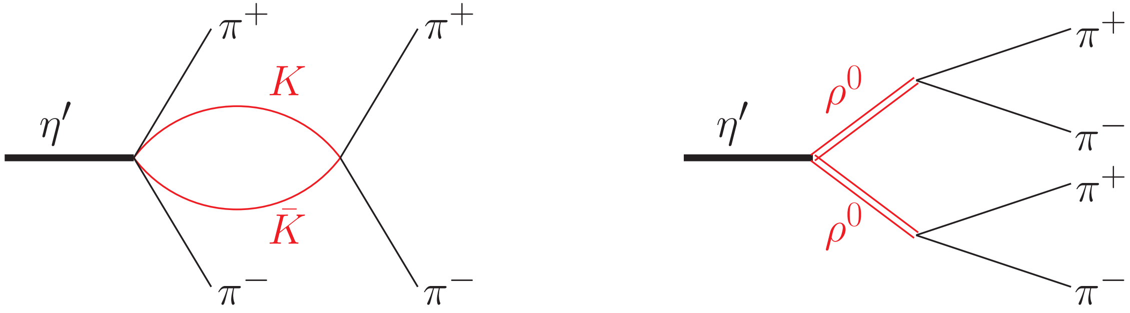

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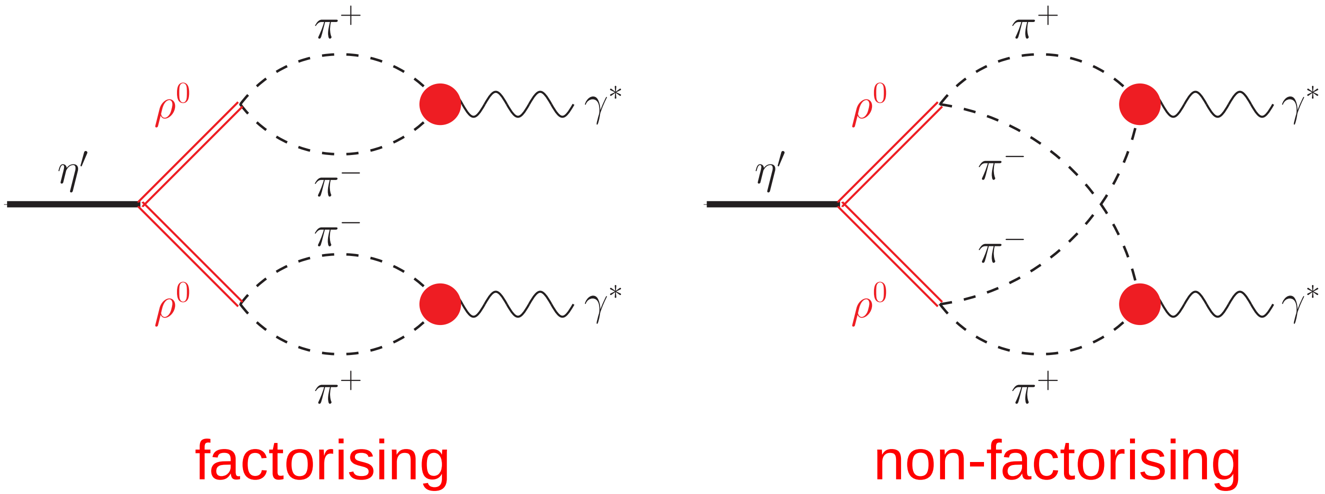
exp: $\mathcal{B}(\eta' \to \pi^+\pi^-\pi^+\pi^-) = (8.5 \pm 0.7 \pm 0.6) \times 10^{-5}$ BESIII 2014

- start analysis of *doubly* virtual η' transition form factor from here?

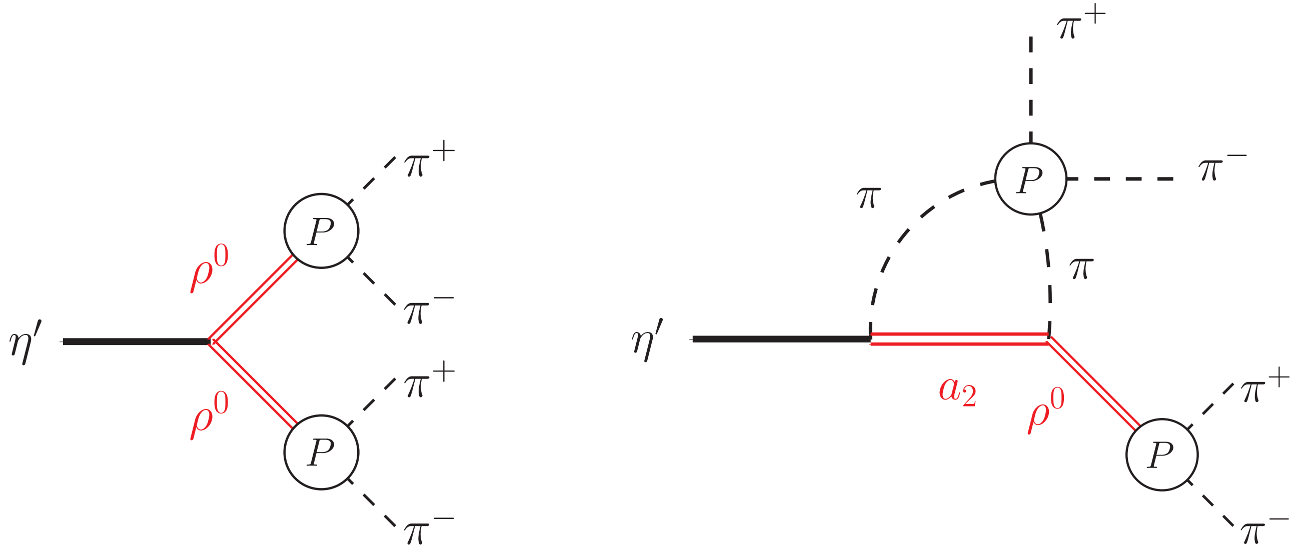


$\longrightarrow$ more differential info on $\eta' \to \pi^+\pi^-\pi^+\pi^-$ highly desirable!

How to go *doubly* virtual? — $\eta' \to \pi^+\pi^-\pi^+\pi^-$

- work in progress: S. Holz, J. Plenter, MSc theses

  implement a_2 left-hand cut for $\eta' \to \pi^+\pi^-\pi^+\pi^-$, too!



- pairwise rescattering only (no Khuri–Treiman for four-body!)
- a_2-exchange provides natural factorisation-breaking
- aim: build double-spectral function for $\eta^{(\prime)}$ transition form factor based on $\eta^{(\prime)} \to \pi^+\pi^-\pi^+\pi^-$ amplitude
- next step: consistency with $\eta' \to \pi^+\pi^-\gamma$ data

Summary / Outlook

Dispersive analyses of $\boldsymbol{\pi^0}$, $\boldsymbol{\eta^{(\prime)}}$ transition form factors:

- high-precision data on

  $e^+e^- \to \pi^+\pi^-\pi^0$ var. / $\eta \to \pi^+\pi^-\gamma$ KLOE / $\eta' \to \pi^+\pi^-\gamma$ BESIII

  allow for high-precision dispersive predictions of π^0, $\eta^{(\prime)} \to \gamma^*\gamma$

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- addition of one effective pole allows to fulfil constraints:

  anomaly Brodsky–Lepage limit pQCD limit

- very preliminary: π^0 pole $(g-2)_\mu^{\pi^0} = (60.2 \pm 3.x) \times 10^{-11}$

  uncertainty: $\pi^0 \to \gamma\gamma$, dispersive uncertainties, BL limit

  $\longrightarrow$ all improvable by singly-virtual experimental input

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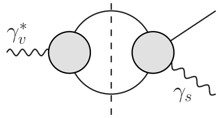
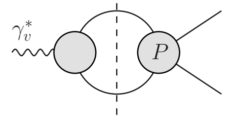
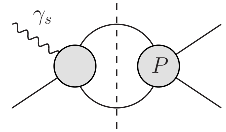
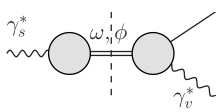
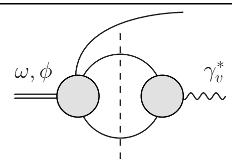
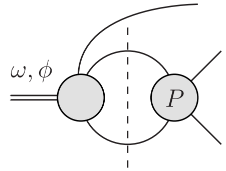
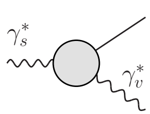
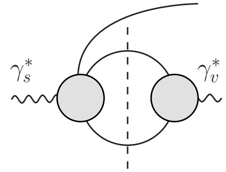
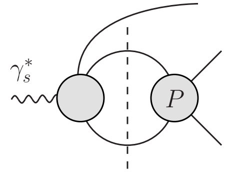
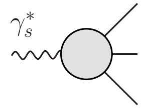
Theoretical work in progress on $\eta^{(\prime)}$:

- improved doubly-virtual $\eta^{(\prime)}$ ($\leftrightarrow \eta^{(\prime)} \to 4\pi$) S. Holz, J. Plenter et al.

Spares

Summary: processes and unitarity relations for $\pi^0 \to \gamma^*\gamma^*$

process	unitarity relations	SC 1	SC 2
			$F_{\pi^0\gamma\gamma}$
		$F_{3\pi}$	$\sigma(\gamma\pi \to \pi\pi)$
			$\Gamma_{\pi^0\gamma}$
		$\Gamma_{3\pi}$	$\frac{d^2\Gamma}{ds\,dt}(\omega,\phi \to 3\pi)$
			$\sigma(e^+e^- \to \pi^0\gamma)$
		$\sigma(e^+e^- \to 3\pi)$	$\sigma(\gamma\pi \to \pi\pi)$ $\frac{d^2\Gamma}{ds\,dt}(\omega,\phi \to 3\pi)$
		$F_{3\pi}$	$\sigma(e^+e^- \to 3\pi)$

Colangelo, Hoferichter, BK, Procura, Stoffer 2014

$\gamma\pi \to \pi\pi$

$\omega \to 3\pi$, $\phi \to 3\pi$

$\gamma^* \to 3\pi$

common theme:
resum $\pi\pi$ rescattering

$\omega/\phi \to 3\pi$: dispersive solution

- identical quantum numbers to $\gamma\pi \to \pi\pi$

$$\mathcal{F}(s) = a\,\Omega(s)\left\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s-i\epsilon)}\right\}$$

$$\hat{\mathcal{F}}(s) = \frac{3}{2}\int_{-1}^{1}dz\,(1-z^2)\mathcal{F}\big(t(s,z)\big)$$

$\longrightarrow$ fix subtraction constant a to partial width(s) $\omega/\phi \to 3\pi$

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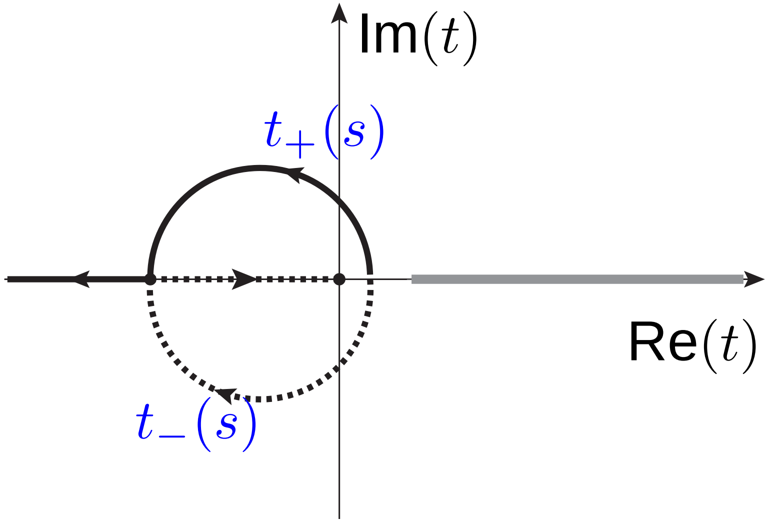
- complication:
  analytic continuation in decay mass M_V required
- $M_V < 3M_\pi$:
  okay



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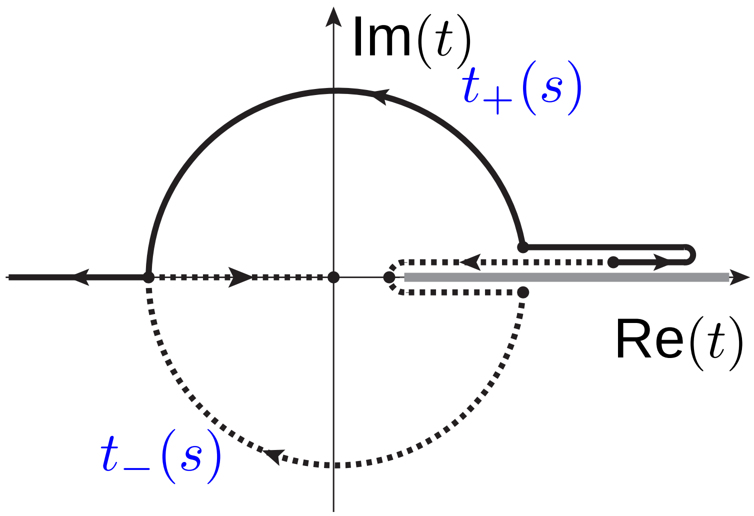

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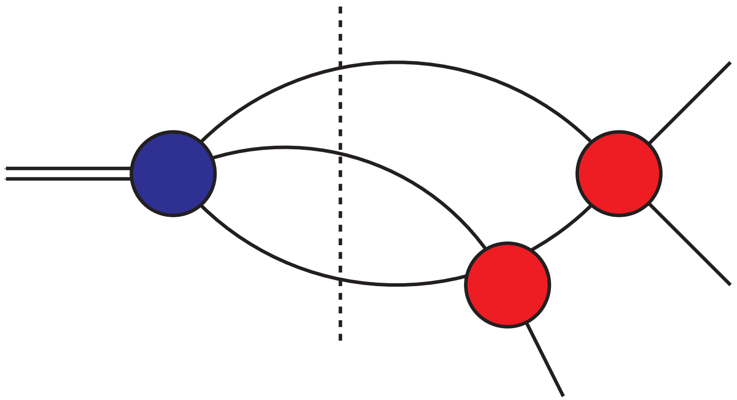
- complication:
  analytic continuation in
  decay mass M_V required
- $M_V > 3M_\pi$:
  path deformation required
  $\longrightarrow$ generates 3-particle cuts

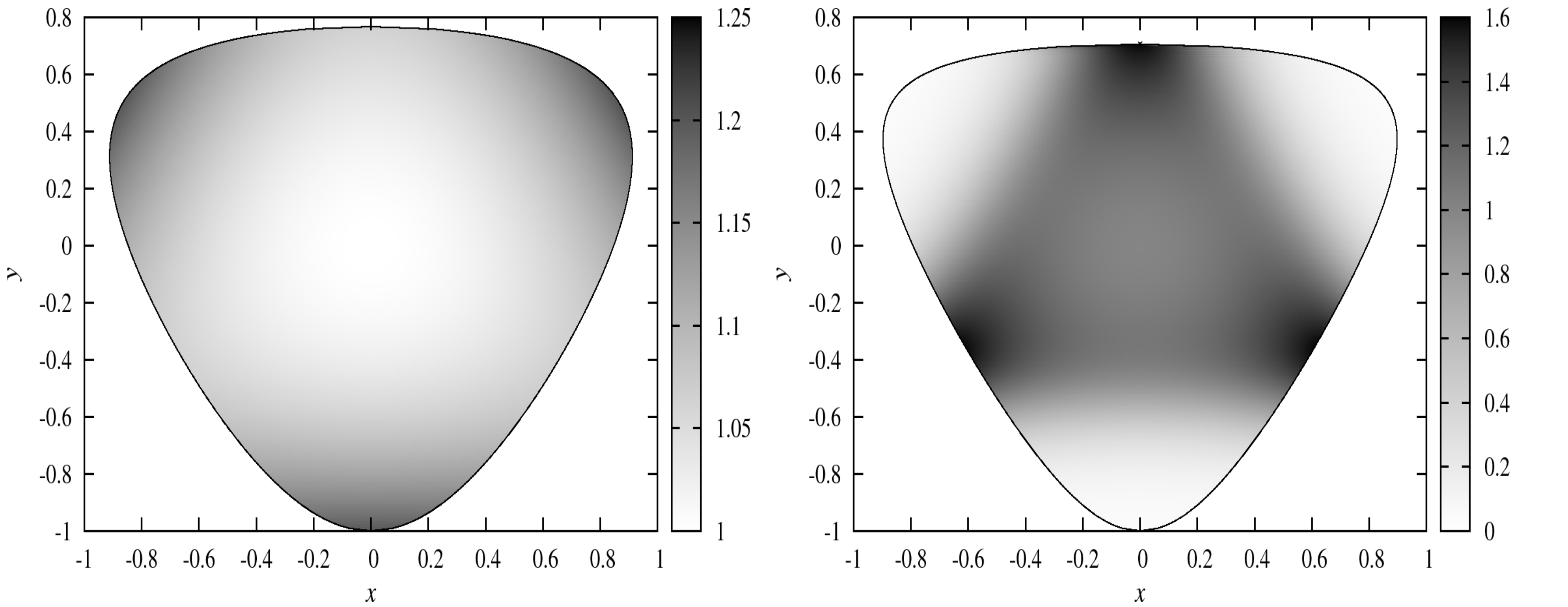


$\omega/\phi \to 3\pi$ Dalitz plots

- subtraction constant a fixed to partial width

  $\longrightarrow$ normalised Dalitz plot a prediction

$\omega \to 3\pi$: $\phi \to 3\pi$:



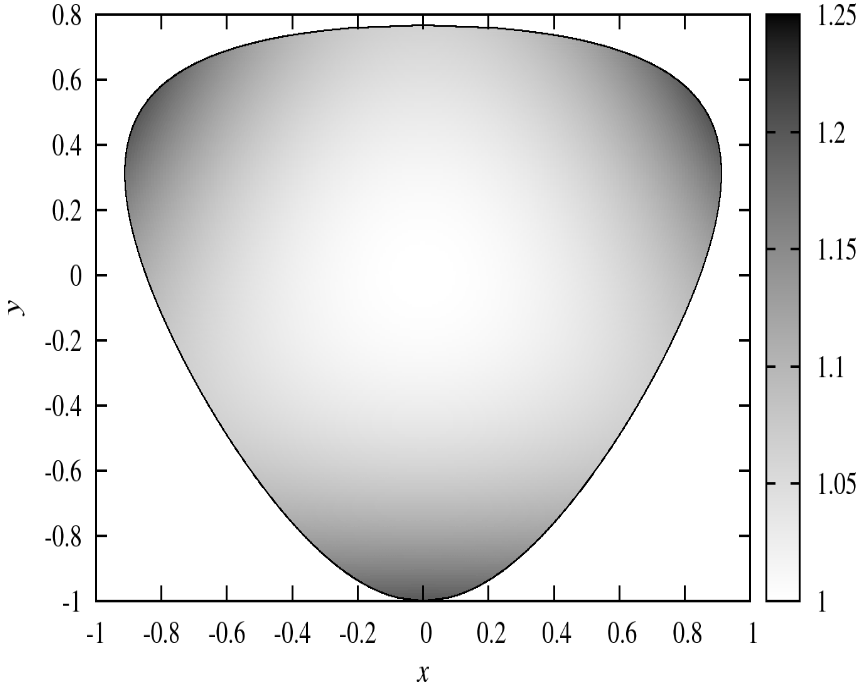
- ω Dalitz plot is relatively smooth
- ϕ Dalitz plot clearly shows ρ resonance bands

Niecknig, BK, Schneider 2012

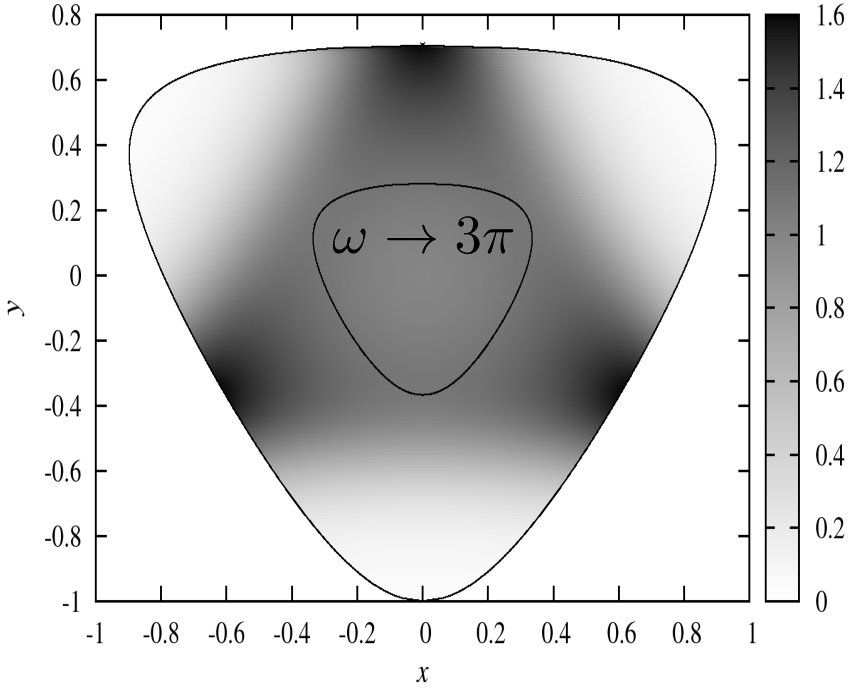
$\omega/\phi \to 3\pi$ Dalitz plots

- subtraction constant a fixed to partial width
  $\longrightarrow$ normalised Dalitz plot a prediction

$\omega \to 3\pi$:



$\phi \to 3\pi$:



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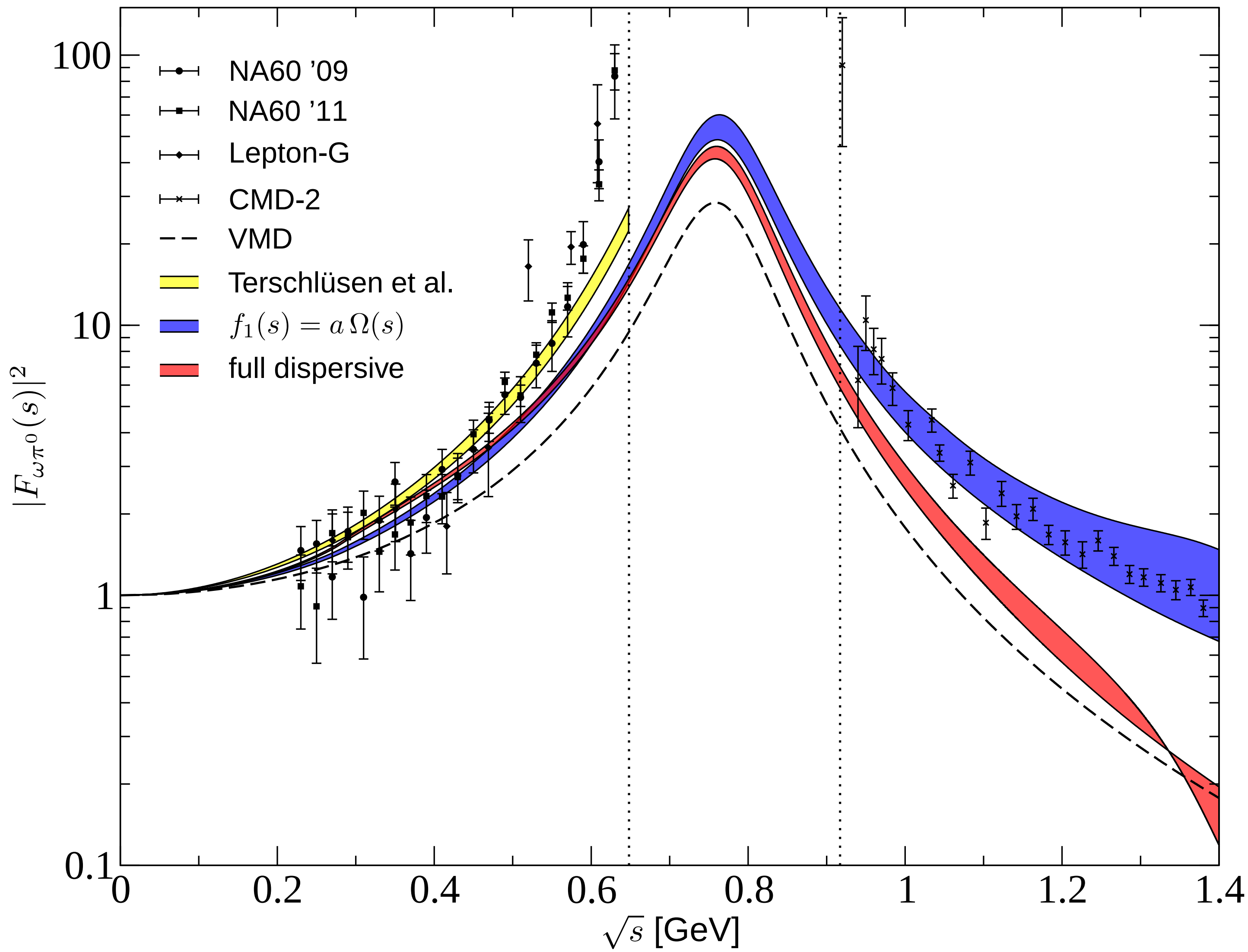
Niecknig, BK, Schneider 2012

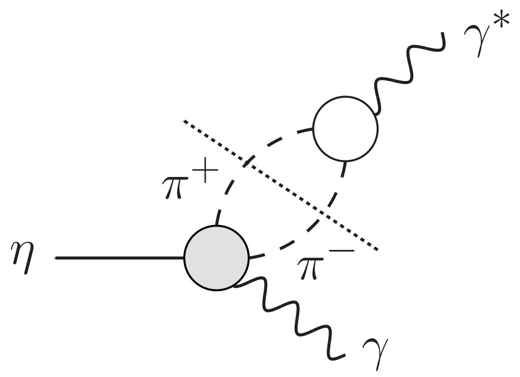
Naive extension to $e^+e^- \to \pi^0\omega$



- full solution above naive VMD, but still too low
- higher intermediate states (4π / $\pi\omega$) more important?

Anomalous decay $\eta \to \pi^+\pi^-\gamma$

- $\alpha_{\text{KLOE}} = (1.52 \pm 0.06)$ GeV^{-2} large
  $\longrightarrow$ implausible to explain through ρ', ρ''...
- for large t, expect $P(t) \to$ const. rather
- $\eta \to \gamma^*\gamma$ transition form factor:
  $\longrightarrow$ dispersion integral covers
  larger energy range



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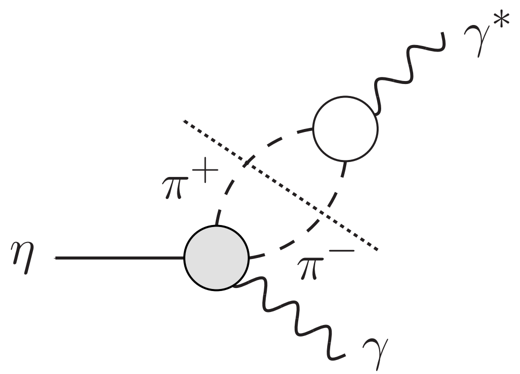


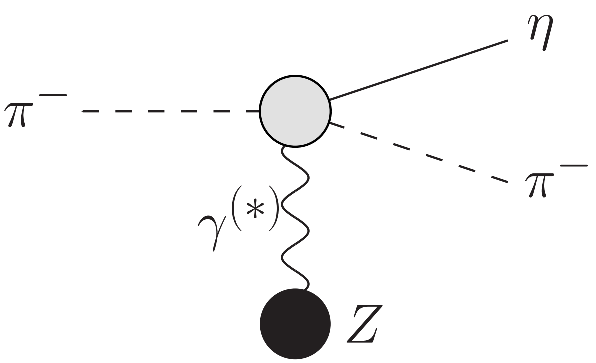
Intriguing observation:

- naive continuation of $\mathcal{F}^{\eta}_{\pi\pi\gamma} = A(1+\alpha t)\Omega(t)$ has zero at $t = -1/\alpha \approx -0.66\,\text{GeV}^2$
  $\longrightarrow$ test this in crossed process $\gamma\pi^- \to \pi^-\eta$
  $\longrightarrow$ "left-hand cuts" in $\pi\eta$ system?

BK, Plenter 2015

Primakoff reaction $\gamma\pi \to \pi\eta$

- can be measured in Primakoff reaction COMPASS
- S-wave forbidden
  P-wave exotic: $J^{PC} = 1^{-+}$
  D-wave $a_2(1320)$ first resonance



Primakoff reaction $\gamma\pi \to \pi\eta$

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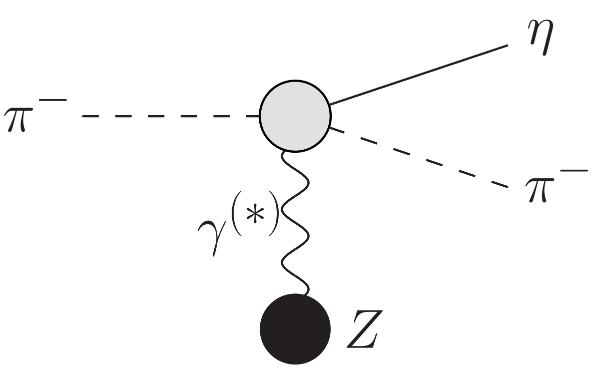
COMPASS

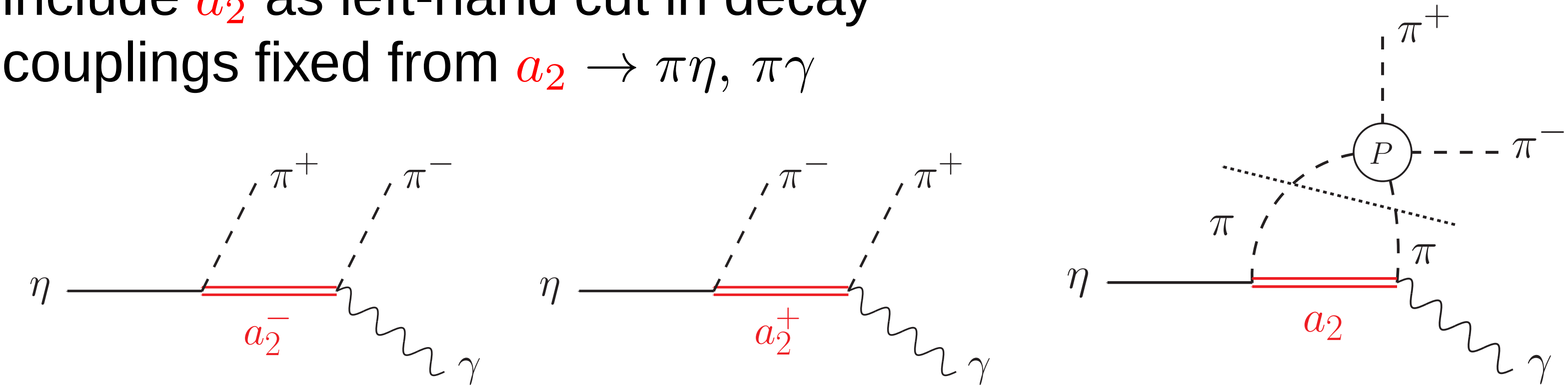


- S-wave forbidden

  P-wave exotic: $J^{PC} = 1^{-+}$

  D-wave $a_2(1320)$ first resonance

- include a_2 as left-hand cut in decay
  couplings fixed from $a_2 \to \pi\eta,\ \pi\gamma$

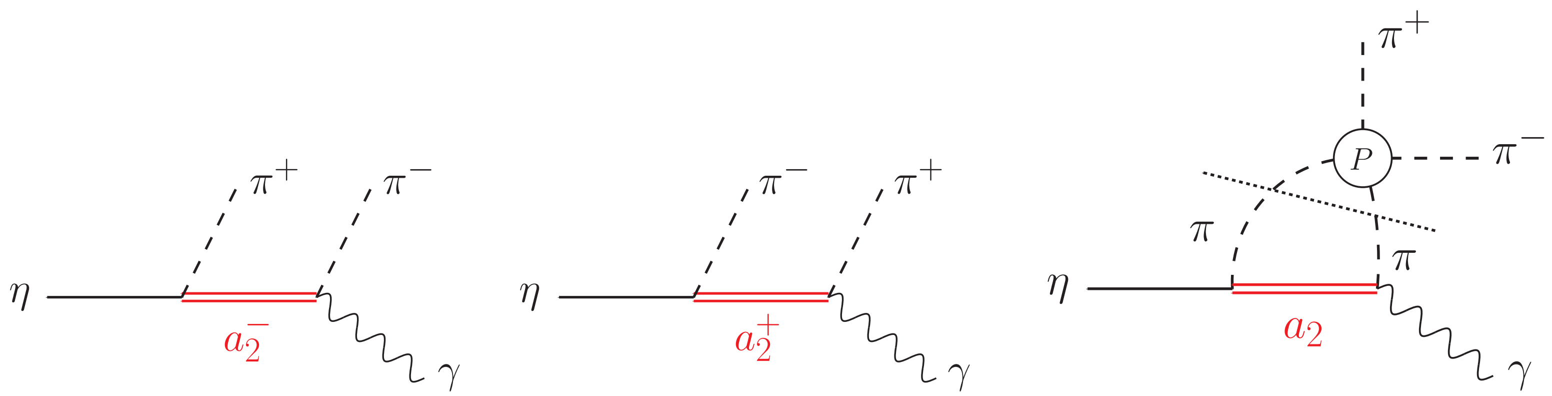


  ▷ compatible with decay data?

  ▷ predictions for $\gamma\pi \to \pi\eta$ cross sections and asymmetries

BK, Plenter 2015

Formalism including left-hand cuts



- a_2 + rescattering essential to preserve Watson's theorem
- formally:

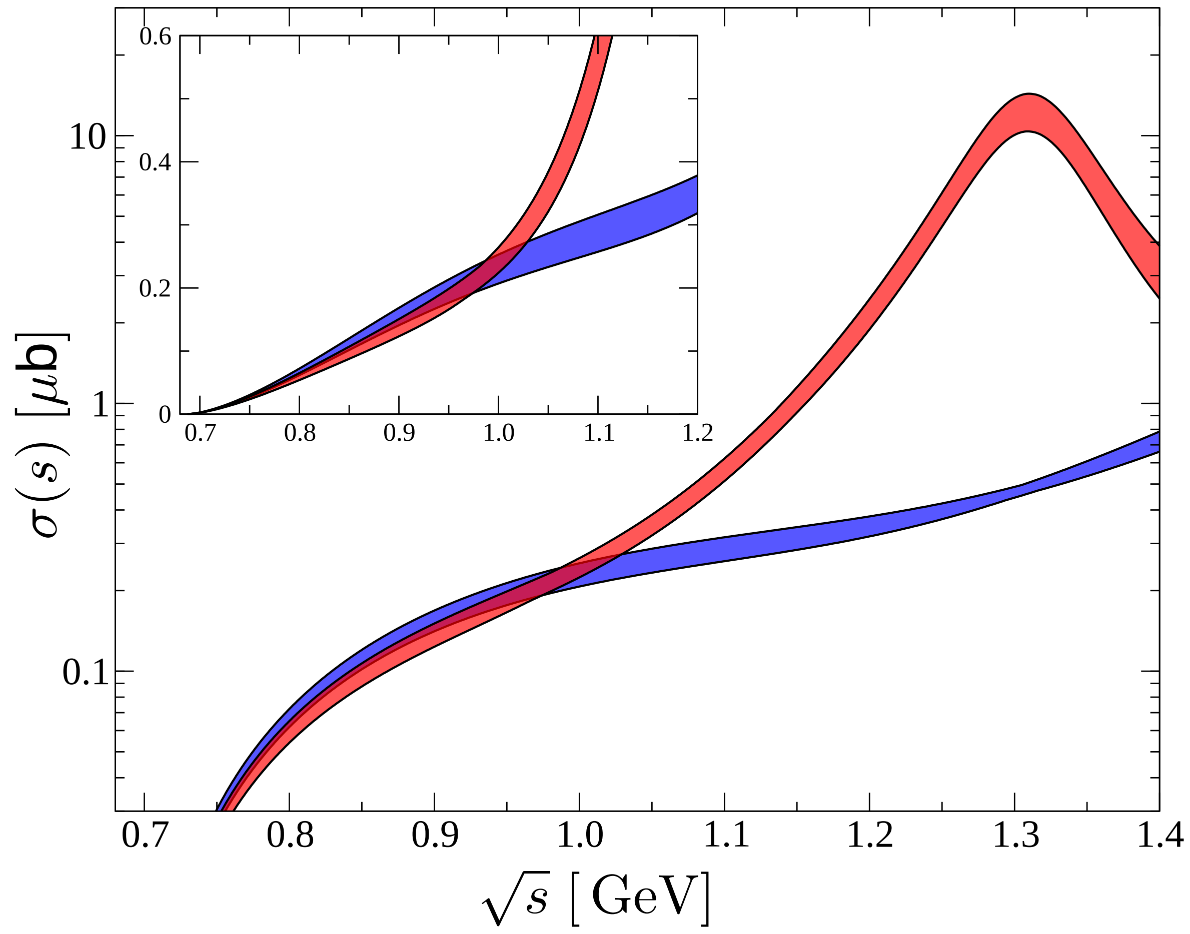
$$\mathcal{F}^{\eta}_{\pi\pi\gamma}(s,t,u) = \mathcal{F}(t) + \mathcal{G}_{a_2}(s,t,u) + \mathcal{G}_{a_2}(u,t,s)$$

$$\mathcal{F}(t) = \Omega(t)\left\{ A(1+\alpha t) + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dx}{x^2}\,\frac{\sin\delta(x)\hat{\mathcal{G}}(x)}{|\Omega(x)|(x-t)}\right\}$$

$\hat{\mathcal{G}}$: t-channel P-wave projection of a_2 exchange graphs

- re-fit subtraction constants A, α

Total cross section $\gamma\pi \to \pi\eta$

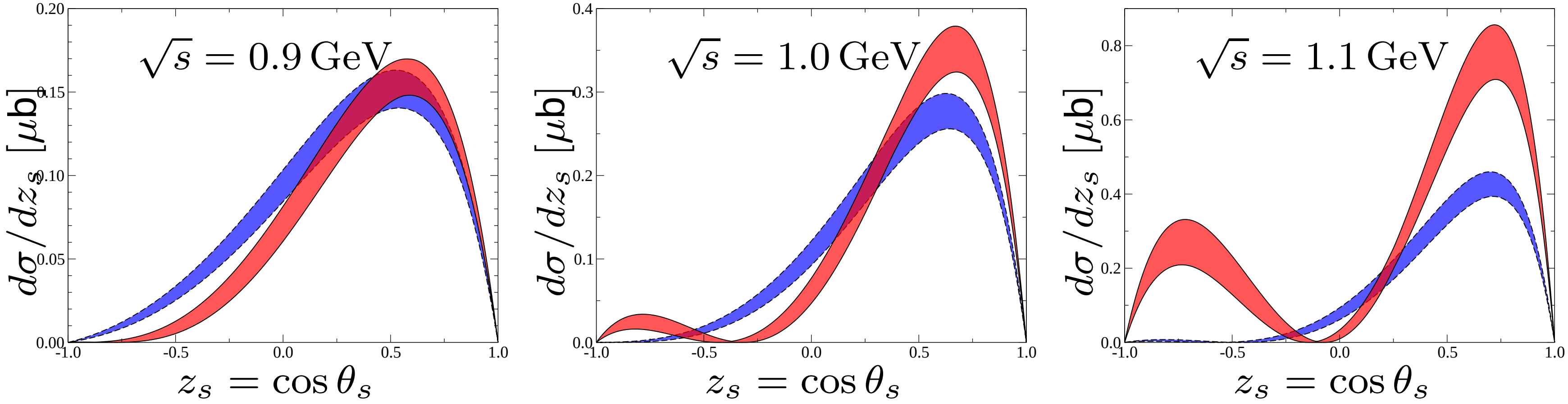


blue: t-channel dynamics / "ρ" only   red: full amplitude

- t-channel dynamics dominate below $\sqrt{s} \approx 1\,\mathrm{GeV}$
- uncertainty bands: $\Gamma(\eta \to \pi^+\pi^-\gamma)$, α, a_2 couplings BK, Plenter 2015

Differential cross sections $\gamma\pi \to \pi\eta$

- amplitude zero visible in differential cross sections:



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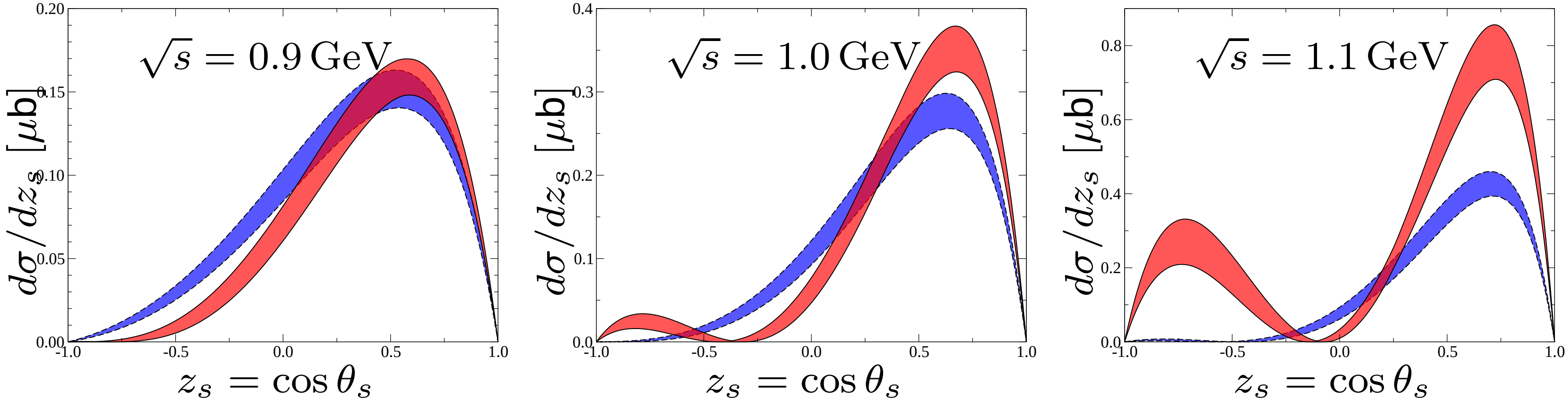

blue: t-channel dynamics / "ρ" only red: full amplitude

- strong P-D-wave interference
- can be expressed as forward-backward asymmetry

$$A_{\mathrm{FB}} = \frac{\sigma(\cos\theta > 0) - \sigma(\cos\theta < 0)}{\sigma_{\mathrm{total}}}$$