Operators

Let \( \hat{D} \equiv \frac{4}{dx} \).

1. Evaluate the following expressions:

   (a) (10 points) \( \hat{D}^{-2} \sqrt{x} \)
   
   (b) (10 points) \( \hat{D}^{-3} \frac{2}{x} \)

   Hint: recall the reasoning that was used in class to derive the following relation:

   \[
   \hat{D}^{-1} f(x) \equiv \int f(x) dx
   \]

2. (15 points) Evaluate the following expression:

   \[
   \left( \frac{1}{(1 + \hat{D})^2} \right) x^2,
   \]

   Hint: use a computer algebra system to find the Taylor series for \( \frac{1}{(1 + x)^2} \). What are the values of the following expressions: \( \hat{D}^3 x^2, \hat{D}^4 x^2, \hat{D}^5 x^2, \ldots \)?

3. (10 points) Recall that the shift operator \( \hat{T}_a \equiv e^{a \hat{D}} \) has the following property:

   \[
   \hat{T}_a f(x) = f(x + a).
   \]

   Which of the expressions below (there may be more than one) are equal to \( f(x + 6a) \)?

   - \( 6 \hat{T}_a f(x) \)
   - \( 3 \hat{T}_{2a} f(x) \)
   - \( 2 \hat{T}_{3a} f(x) \)
   - \( \hat{T}_{6a} f(x) \)
   - \( \hat{T}_{6a}^6 f(x) \)
   - \( \hat{T}_{2a}^3 f(x) \)
   - \( \hat{T}_{3a}^2 f(x) \)
   - All of the above
   - None of the above
Logarithmic differentiation

4. (10 points) Find the derivative \( \frac{d}{dx} x^{(e^x)} \)

Euler-MacLaurin summation formula

5. Recall that

\[
\sum_{n=m}^{N+m} f(n) = \int_{m}^{m+N} f(x) dx + \frac{1}{2} (f(N+m) + f(m)) + \frac{1}{12} (f'(N+m) - f'(m)) - \frac{1}{720} (f'''(N+m) - f'''(m)) + \ldots \quad (1)
\]

Consider the following sum:

\[
S = \sum_{n=2}^{\infty} \frac{1}{n \log(n)^{3/2}}. \quad (2)
\]

It is a particular case of Eq. (1) for \( m = 2, N = \infty, f(x) = \frac{1}{x \log(x)^{3/2}}. \)

(a) (15 points) Evaluate the sum \( S \), Eq. (2), using the first two terms in Euler-MacLaurin formula, Eq. (1)

(b) (10 points) Estimate how long it would take to calculate the sum \( S \) with the absolute error \( \delta \sim \frac{1}{10} \) naively summing terms in Eq. (2) on a computer capable of adding \( 10^{15} \) terms per second.\(^1\)

6. (15 points) Evaluate the following product:

\[
\alpha = \prod_{n=1}^{\infty} n^{1/n^2}. \quad (3)
\]

Hint: Use Euler-MacLaurin summation formula, Eq. (1) to evaluate \( \log(\alpha) \). Use only the integral term of the expression.

\(^1\)Such a computer is “slightly” faster than the fastest supercomputer available as of November 2012 - see www.top500.org