Physics 1402: Lecture 24
Today’s Agenda

• Announcements:
  – Herschbach: Nobel Prize winner speaking Thursday
    » 4:00PM in bio-physics building (BP-130 ?)

• Midterm 2: NOT Nov. 6
  – Following week ...

• Homework 07: due Friday next week

• AC current
  – RLC circuits
  – Phasers
  – Resonances
Lecture 4

Power Production
An Application of Faraday’s Law

• A design schematic

\[ \varepsilon = -\frac{d}{dt} BA \cos \omega t = \omega BA \sin \omega t \]

AC Circuits
Series LCR

• Statement of problem:
  Given \( \varepsilon = \varepsilon_m \sin \omega t \), find \( i(t) \).
  Everything else will follow.

• Procedure: start with loop equation?

\[ L \frac{d^2 Q}{dt^2} + \frac{Q}{C} + R \frac{dQ}{dt} = \varepsilon_m \sin \omega t \]

• We could solve this equation in the same manner we did for the LCR damped circuit. Rather than slog through the algebra, we will take a different approach which uses phasors.
Lecture 4

Phasors

- **R**: \( V \) in phase with \( i \)
  \[ V_R = R_i = V_m \sin \omega t \]

- **C**: \( V \) lags \( i \) by 90°
  \[ V_C = \frac{Q}{C} = V_m \sin \omega t \]

- **L**: \( V \) leads \( i \) by 90°
  \[ V_L = L \frac{di}{dt} = V_m \sin \omega t \]

A phasor is a vector whose magnitude is the maximum value of a quantity (e.g. \( V \) or \( I \)) and which rotates counterclockwise in a 2-d plane with angular velocity \( \omega \). Recall uniform circular motion:

The projections of \( r \) (on the vertical y axis) execute sinusoidal oscillation.

\[
\begin{align*}
  x &= r \cos \omega t \\
  y &= r \sin \omega t
\end{align*}
\]

Suppose:

- \( i = i_m \sin \omega t \)
- \( V_R = R_i_m \sin \omega t \)
- \( V_C = -\frac{1}{\omega C} i_m \cos \omega t \)
- \( V_L = \omega L i_m \cos \omega t \)
Lecture 24, ACT 1

- A series LCR circuit driven by emf \( \varepsilon = \varepsilon_0 \sin \omega t \) produces a current \( i = i_m \sin(\omega t - \phi) \). The phasor diagram for the current at \( t=0 \) is shown to the right.
  - At which of the following times is \( V_C \), the magnitude of the voltage across the capacitor, a maximum?

\[
(a) \quad t=0 \\
(b) \quad t=t_b \\
(c) \quad t=t_c
\]

Series LCR AC Circuit

- Consider the circuit shown here: the loop equation gives:

\[
L \frac{d^2 Q}{dt^2} + \frac{Q}{C} + R \frac{dQ}{dt} = \varepsilon_m \sin \omega t
\]

- Assume a solution of the form:

\[
i = i_m \sin(\omega t - \phi)
\]

- Here all unknowns, \( (i_m, \phi) \), must be found from the loop eqn; the initial conditions have been taken care of by taking the emf to be: \( \varepsilon = \varepsilon_m \sin \omega t \).

- To solve this problem graphically, first write down expressions for the voltages across R, C, and L and then plot the appropriate phasor diagram.
Lecture 4

Phasors: LCR

- **Given:** \( \varepsilon = \varepsilon_m \sin \omega t \)
- **Assume:**
  - \( i = i_m \sin(\omega t - \phi) \)
  - \[ Q = - \frac{i_m}{\omega} \cos(\omega t - \phi) \]
  - \[ \frac{di}{dt} = i_m \omega \cos(\omega t - \phi) \]

\[ \begin{align*}
V_R &= Ri = R i_m \sin(\omega t - \phi) \\
V_C &= \frac{Q}{C} = -\frac{1}{\omega C} i_m \cos(\omega t - \phi) \\
V_L &= L \frac{di}{dt} = \omega L i_m \cos(\omega t - \phi)
\end{align*} \]

- From these equations, we can draw the phasor diagram to the right.
- This picture corresponds to a snapshot at \( t=0 \). The projections of these phasors along the vertical axis are the actual values of the voltages at the given time.

Phasors: LCR

- The phasor diagram has been relabeled in terms of the reactances defined from:
  - \( X_L \equiv \omega L \)
  - \( X_C \equiv \frac{1}{\omega C} \)

The unknowns \((i_m, \phi)\) can now be solved for graphically since the vector sum of the voltages \( V_L + V_C + V_R \) must sum to the driving emf \( \varepsilon \).
Lecture 4

Phasors: LCR

\[ X_L \equiv \omega L \]
\[ X_C \equiv \frac{1}{\omega C} \]
\[ Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \]

\[ \tan \phi = \frac{X_L - X_C}{R} \]
\[ \varepsilon_m = \varepsilon_m [R^2 + (X_L - X_C)^2] \]
\[ i_m = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\varepsilon_m}{Z} \]

### Impedance Values and Phase Angles for Various Circuit-Element Combinations

<table>
<thead>
<tr>
<th>Circuit Elements</th>
<th>Impedance Z</th>
<th>Phase Angle ( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>( R )</td>
<td>( 0^\circ )</td>
</tr>
<tr>
<td>( L )</td>
<td>( X_L )</td>
<td>( -90^\circ )</td>
</tr>
<tr>
<td>( C )</td>
<td>( X_C )</td>
<td>( +90^\circ )</td>
</tr>
<tr>
<td>( R )</td>
<td></td>
<td>( L )</td>
</tr>
<tr>
<td>( R )</td>
<td></td>
<td>( L )</td>
</tr>
</tbody>
</table>
| \( R \) || \( L \) | \( C \) | \( \sqrt{R^2 + (X_L - X_C)^2} \) | Negative if \( X_C > X_L \)
|                |             | Positive if \( X_C < X_L \) |

*In each case, an AC voltage (not shown) is applied across the elements.*
**Phasors: Tips**

- This phasor diagram was drawn as a snapshot of time \( t=0 \) with the voltages being given as the projections along the \( y \)-axis.
- Sometimes, in working problems, it is easier to draw the diagram at a time when the current is along the \( x \)-axis (when \( i=0 \)).

From this diagram, we can also create a triangle which allows us to calculate the impedance \( Z \):

**Resonance**

- For fixed \( R, C, L \) the current \( i_m \) will be a maximum at the resonant frequency \( \omega_0 \) which makes the impedance \( Z \) purely resistive.

\[
\begin{align*}
\text{ie: } i_m &= \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\varepsilon_m}{Z} \\
\text{reaches a maximum when: } X_L &= X_C
\end{align*}
\]

the frequency at which this condition is obtained is given from:

\[
\omega_0 L = \frac{1}{\omega_0 C} \quad \Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}
\]

- Note that this resonant frequency is identical to the natural frequency of the LC circuit by itself!
- At this frequency, the current and the driving voltage are in phase:

\[
\tan \phi = \frac{X_L - X_C}{R} = 0
\]
Resonance

The current in an LCR circuit depends on the values of the elements and on the driving frequency through the relation

\[ i_m = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\varepsilon_m}{Z} \]

Suppose you plot the current versus \( \omega \), the source voltage frequency, you would get:

“Impedance Triangle”

Power in LCR Circuit

- The power supplied by the \( emf \) in a series LCR circuit depends on the frequency \( \omega \). It will turn out that the maximum power is supplied at the resonant frequency \( \omega_0 \).

- The instantaneous power (for some frequency, \( \omega \)) delivered at time \( t \) is given by:

\[ P(t) = \varepsilon(t)i(t) = [\varepsilon_m \sin \omega t] \cdot [i_m \sin(\omega t - \phi)] \]

- The most useful quantity to consider here is not the instantaneous power but rather the average power delivered in a cycle.

\[ \langle P(t) \rangle = \varepsilon_m i_m \langle \sin \omega t \cdot \sin(\omega t - \phi) \rangle \]

- To evaluate the average on the right, we first expand the \( \sin(\omega t - \phi) \) term.
Power in LCR Circuit

- Expanding,
\[ \sin \omega t \cdot \sin(\omega t - \phi) = \sin \omega t \cdot (\sin \omega t \cos \phi - \cos \omega t \sin \phi) \]

- Taking the averages,
\[ \langle \sin \omega t \cdot \cos \omega t \rangle = 0 \]
(Integral of Product of even and odd function = 0)

- Generally:
\[ \langle \sin^2 x \rangle = \frac{1}{2\pi} \int_0^{2\pi} dx \sin^2 x = \frac{1}{2} \]

- Putting it all back together again,
\[ \langle P(t) \rangle = \varepsilon_m i_m \left\{ \cos \phi \langle \sin^2 \omega t \rangle - \sin \phi \langle \sin \omega t \cdot \cos \omega t \rangle \right\} \]
\[ \langle P(t) \rangle = \frac{1}{2} \varepsilon_m i_m \cos \phi \]

Power in LCR Circuit

- This result is often rewritten in terms of rms values:
\[ \varepsilon_{\text{rms}} \equiv \frac{1}{\sqrt{2}} \varepsilon_m \quad i_{\text{rms}} \equiv \frac{1}{\sqrt{2}} i_m \quad \Rightarrow \langle P(t) \rangle = \varepsilon_{\text{rms}} i_{\text{rms}} \cos \phi \]

- Power delivered depends on the phase, \( \phi \), the "power factor"
- phase depends on the values of L, C, R, and \( \omega \)
- therefore...
Lecture 4

**Power and Resonance in RLC**

\[ \langle P(t) \rangle = \varepsilon_{\text{rms}} i_{\text{rms}} \cos \phi \]

- Power, as well as current, peaks at \( \omega = \omega_0 \). The sharpness of the resonance depends on the values of the components.
- Recall:
  \[ i_m = \frac{\varepsilon_m}{R} \left( \frac{R}{Z} \right) = \frac{\varepsilon_m}{R} \cos \phi \]
  \[ R \]
  \[ \phi \]
  \[ Z \]
  \[ |X_L - X_C| \]

- Therefore,
  \[ \langle P(t) \rangle = \frac{\varepsilon_{\text{rms}}^2}{R} \cos^2 \phi = \left( \frac{\varepsilon_{\text{rms}}^2}{R} \right) \frac{R^2}{R^2 + (X_L - X_C)^2} \]

We can write this in the following manner (which we won’t try to prove):

\[ \langle P(t) \rangle = \frac{\varepsilon_{\text{rms}}^2}{R} \frac{x^2}{x^2 + Q^2(x^2 - 1)^2} \]

...introducing the curious factors \( Q \) and \( x \)

\[ x \equiv \frac{\omega}{\omega_{\text{res}}} \quad Q \equiv \frac{\omega_{\text{res}} L}{R} \]

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**The Q factor**

A parameter “\( Q \)” is often defined to describe the sharpness of resonance peaks in both mechanical and electrical oscillating systems. “\( Q \)” is defined as

\[ Q \equiv 2\pi \frac{U_{\text{max}}}{\Delta U} \]

where \( U_{\text{max}} \) is max energy stored in the system and \( \Delta U \) is the energy dissipated in one cycle.

For RLC circuit, \( U_{\text{max}} \) is (e.g.)

\[ U_{\text{max}} = \frac{1}{2} L I_{\text{max}}^2 \]

And losses only in \( R \), namely

\[ \Delta U = \frac{1}{2} I_{\text{max}}^2 R T = \frac{1}{2} I_{\text{max}}^2 R \left( \frac{2\pi}{\omega_{\text{res}}} \right) \]

This gives

\[ Q \equiv \frac{\omega_{\text{res}} L}{R} \]
The Q factor

Q also determines the sharpness of the resonance peaks in a graph of Power delivered by the source versus frequency.

The average power in our LCR circuit is given by

\[ P = \left( \frac{\varepsilon_{\text{rms}}}{R} \right)^2 \frac{R^2 \omega^2}{R^2 \omega^2 + L^2 (\omega^2 - \omega_0^2)^2} \]

We know this has a maximum at \( \omega = \omega_0 \).

It turns out (exercise for you) that this function reaches \( \frac{1}{2} \) of its maximum at the values,

\[ \omega = \omega_0 \pm \frac{1}{2} \frac{R}{L} \]

This gives a new meaning to Q.

\[ Q = \frac{\omega_0 L}{R} = \frac{\omega_0}{\Delta \omega} \]
Lecture 24, ACT 2

- Consider the two circuits shown where \( C_{\text{II}} = 2 C_{\text{I}} \).
  - What is the relation between the quality factors, \( Q_{\text{I}} \) and \( Q_{\text{II}} \), of the two circuits?
    
    (a) \( Q_{\text{II}} < Q_{\text{I}} \)  
    (b) \( Q_{\text{II}} = Q_{\text{I}} \)  
    (c) \( Q_{\text{II}} > Q_{\text{I}} \)

Lecture 24, ACT 3

- Consider the two circuits shown where \( C_{\text{II}} = 2 C_{\text{I}} \) and \( L_{\text{II}} = \frac{1}{2} L_{\text{I}} \).
  - Which circuit has the narrowest width of the resonance peak?
    
    (a) I  
    (b) II  
    (c) Both the same
Lecture 4

Power Transmission

- How do we transport power from power stations to homes?
  - At home, the AC voltage obtained from outlets in this country is 120V at 60Hz.
  - Transmission of power is typically at very high voltages (~500 kV) (a “high tension” line)
  - Transformers are used to raise the voltage for transmission and lower the voltage for use. We’ll describe these next.

- But why?
  - Calculate ohmic losses in the transmission lines:
  - Define efficiency of transmission:

\[
\text{eff} \equiv \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{i V_{\text{in}} - i^2 R}{i V_{\text{in}}} = 1 - \frac{i R}{V_{\text{in}}} \left( \frac{V_{\text{in}}}{V_{\text{in}}} \right) = 1 - \frac{P_{\text{in}} R}{V_{\text{in}}^2}
\]

- Note for fixed input power and line resistance, the inefficiency \( \propto 1/V^2 \)

Transformers

- AC voltages can be stepped up or stepped down by the use of transformers.

- The AC current in the primary circuit creates a time-varying magnetic field in the iron

- This induces an emf on the secondary windings due to the mutual inductance of the two sets of coils.

- The iron is used to maximize the mutual inductance. We assume that the entire flux produced by each turn of the primary is trapped in the iron.
Lecture 4

**Ideal Transformers (no load)**

- The primary circuit is just an AC voltage source in series with an inductor. The change in flux produced in each turn is given by:
  \[
  \frac{d\phi_{\text{turn}}}{dt} = \frac{V_1}{N_1}
  \]

- The change in flux per turn in the secondary coil is the same as the change in flux per turn in the primary coil (ideal case). The induced voltage appearing across the secondary coil is given by:
  \[
  V_2 = N_2 \frac{d\phi_{\text{turn}}}{dt} = \frac{N_2}{N_1} V_1
  \]

- Therefore,
  - \(N_2 > N_1\) \(\Rightarrow\) secondary \(V_2\) is larger than primary \(V_1\) (step-up)
  - \(N_1 > N_2\) \(\Rightarrow\) secondary \(V_2\) is smaller than primary \(V_1\) (step-down)

- Note: “no load” means no current in secondary. The primary current, termed “the magnetizing current” is small!

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**Ideal Transformers with a Load**

- What happens when we connect a resistive load to the secondary coil?
  - Flux produced by primary coil induces an *emf* in secondary
  - *emf* in secondary produces current \(i_2\)
    \[
    i_2 = \frac{V_2}{R}
    \]
  - This current produces a flux in the secondary coil \(\propto N_2 i_2\), which opposes the original flux -- Lenz’s law
  - This changing flux appears in the primary circuit as well; the sense of it is to reduce the *emf* in the primary...
  - However, \(V_1\) is a voltage source.
  - Therefore, there must be an increased current \(i_1\) (supplied by the voltage source) in the primary which produces a flux \(\propto N_1 i_1\), which exactly cancels the flux produced by \(i_2\).
Transformers with a Load

- With a resistive load in the secondary, the primary current is given by:

\[ i_1 = \frac{N_2}{N_1} i_2 = \frac{N_2 V_2}{N_1} R = \frac{V_1}{(N_1/N_2)^2 R} \]

Lecture 24, ACT 4

- The primary coil of an ideal transformer is connected to a battery \((V_1 = 12V)\) as shown. The secondary winding has a load of \(2 \Omega\). There are 50 turns in the primary and 200 turns in the secondary.

What is the current in the secondary?

(a) 24 A  
(b) 1.5 A  
(c) 6 A  
(d) 0 A
• The primary coil of an ideal transformer is connected to the wall ($V_1 = 120V$) as shown. There are 50 turns in the primary and 200 turns in the secondary.

– If 960 W are dissipated in the resistor $R$, what is the current in the primary?

(a) 8 A  
(b) 16 A  
(c) 32 A

Fields from Circuits?

• We have been focusing on what happens within the circuits we have been studying (eg currents, voltages, etc.)

• What's happening outside the circuits??
  – We know that:
    » charges create electric fields and
    » moving charges (currents) create magnetic fields.
  – Can we detect these fields?
  – Demos:
    » We saw a bulb connected to a loop glow when the loop came near a solenoidal magnet.
    » Light spreads out and makes interference patterns.
    Do we understand this?