Complex numbers

1. (10 points) Find the coordinate and the polar form of the following complex number:

\[ Z = \left( \frac{\sqrt{2} - i\sqrt{2}}{1 - i\sqrt{3}} \right)^{26}. \]

Answer: \( Z = e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2} \)

2. (10 points) Find the values of \( Z = (\sqrt{i})^i. \)

Answer: \( Z = e^{-\frac{\pi}{4} - \pi n}, \) where \( n = 0, \pm 1, \pm 2, \ldots \)

3. (10 points) Find the coordinate and the polar forms of the solutions of the equation:

\[ z^4 = \sqrt{3} - i. \]

How many roots are there?

Cauchy-Riemann equations

4. (10 points) Use Cauchy-Riemann equations to find the analytic function \( f(z), z = x + iy, \) such that its real part is as following:

\[ \text{Re} \ f(z) = u(x, y) = e^x \sin y, \]

and

\[ f(i\pi) = 0. \]

Express the result for \( f(z) \) as a function of \( z \) only.

Answer: \( f(z) = -i(e^z + 1). \)
The Cauchy integral theorem

5. (25 points) Evaluate the integral

\[ I = \int_0^\infty \sin(x^3) \, dx \]

Hints: consider the integral

\[ \oint_{C} e^{-z^3} \, dz \]

along the contour \( C \) sketched in Fig. 1; use the Euler formula; use the fact that

\[ \int_0^\infty e^{-x^3} \, dx \equiv \Gamma\left(\frac{4}{3}\right), \]

where \( \Gamma \) is gamma function. (Can you show this?)

Figure 1: Integration contour for Problem 5. \( (R \to \infty) \).

Answer: \( I = \frac{1}{2} \Gamma\left(\frac{4}{3}\right) \).